Linnea: Automatic Generation of Efficient Linear Algebra Programs

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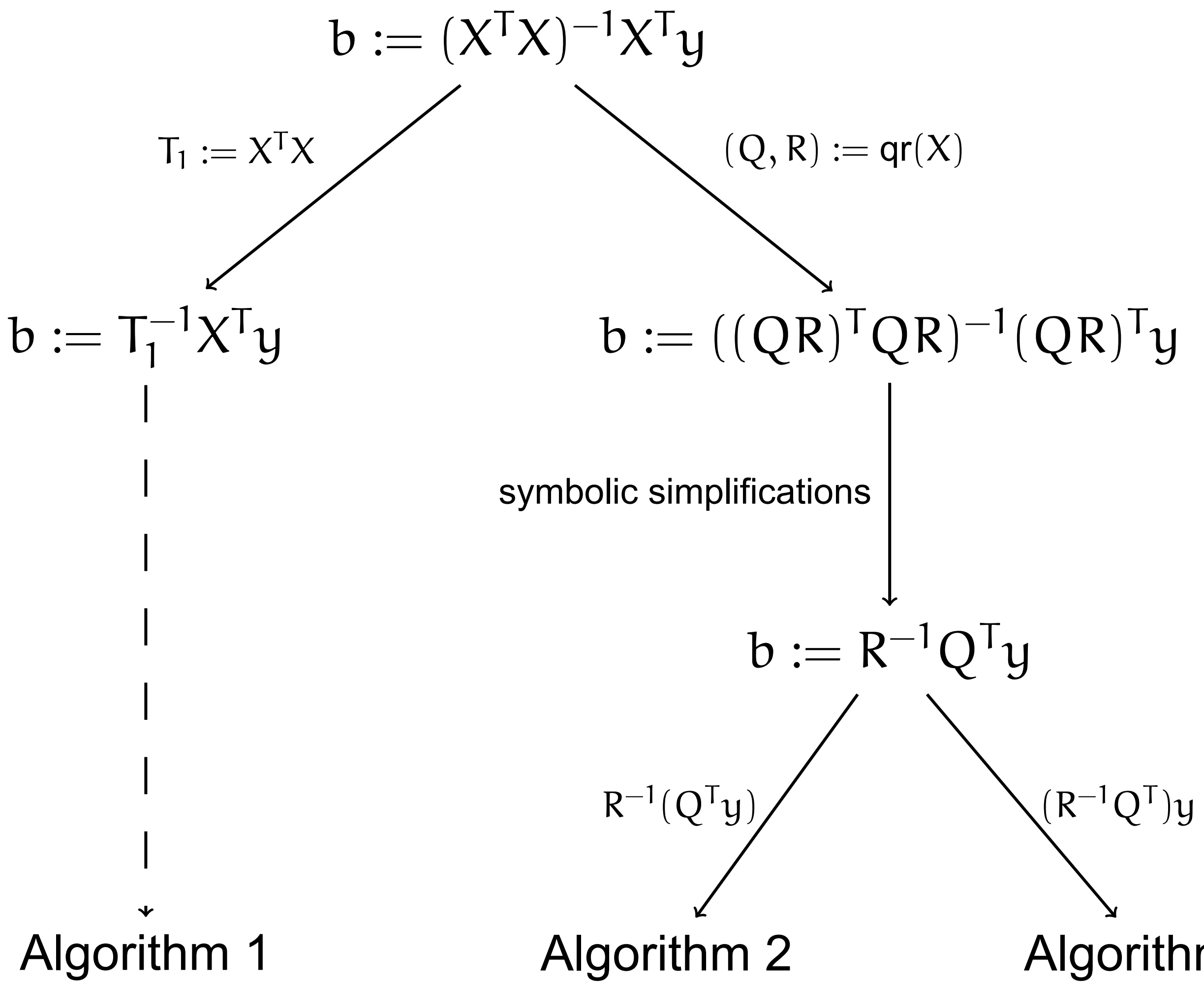
Introduction

- How to compute the following expressions?

$$
\begin{aligned}
b &:= (X^T X)^{-1} X^T y \\
b &:= (X^T M^{-1} X)^{-1} X^T M^{-1} y \\
x &:= W(A^T (A W A^T)^{-1} b - c) \\
x &:= \left(A^{-T} B^T B A^{-1} + R^T [\Lambda(Rz)] R\right)^{-1} A^{-T} B^T B A^{-1} y
\end{aligned}
$$

- High-level languages are easy to use, but performance is usually suboptimal.
- BLAS and LAPACK can be fast, but require a lot of expertise.
- Goal: Simplicity **and** performance.

Introduction



Input

```
n = 1500
m = 1000

Matrix M(n, n) <Input, SPD>
Matrix X(n, m) <Input, FullRank>
ColumnVector y(n) <Input>
ColumnVector b(m) <Output>

b = inv(X'*inv(M)*X)*X'*inv(M)*y
```

Instruction Set

BLAS [DDC$^+$90]

- $y \leftarrow Ax + y$
- $C \leftarrow AB + C$
- $B \leftarrow A^{-1}B$
- $\dots$

LAPACK [AB$^+$99]

- Matrix factorizations.
- Eigensolvers.
- Solvers for linear systems with specific properties.

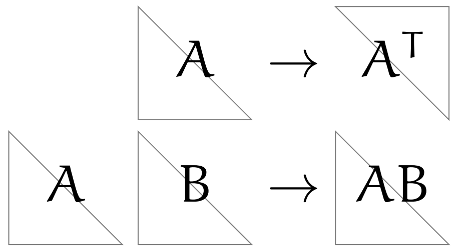
Linear Algebra Knowledge

Properties

Operation	Cost
$A^{-1} B$	n^3
$A^{-1} B$	$2.7n^3$

Linear Algebra Knowledge

Inference of Properties



$$\text{expr} = \text{expr}^T \rightarrow \text{Symmetric}(\text{expr})$$

Linear Algebra Knowledge

Simplifications

$$(AB)^T \rightarrow B^T A^T$$

$$A^T \rightarrow A \qquad \text{if Symmetric}(A)$$

$$Q^T Q \rightarrow I \qquad \text{if Orthogonal}(Q)$$

Rewriting Expressions

$$X := AB + AC \quad \rightarrow \quad X := A(B + C)$$

$$X := A^T A + A^T B + B^T A \quad \rightarrow \quad \begin{array}{l} Y := B + A/2 \\ X := A^T Y + Y^T A \end{array}$$

$$X := (\alpha I + Q^T W Q) \quad \rightarrow \quad \begin{array}{l} Y := \alpha I + W \\ X := Q^T Y Q \end{array}$$

Common Subexpression Elimination

$$\begin{aligned} X &:= AB^{-T}C \\ Y &:= B^{-1}A^{T}C \end{aligned} \quad \rightarrow \quad \begin{aligned} Z &:= AB^{-T} \\ X &:= ZC \\ Y &:= Z^{T}C \end{aligned}$$

$$AB^{-T} = \left(B^{-1}A^{T}\right)^{T} = Z$$

Linear Algebra Knowledge

Generalized Matrix Chain Problem

$$A \; B \; c$$

$(AB)c \quad \mathcal{O}(n^3)$

$A(Bc) \quad \mathcal{O}(n^2)$

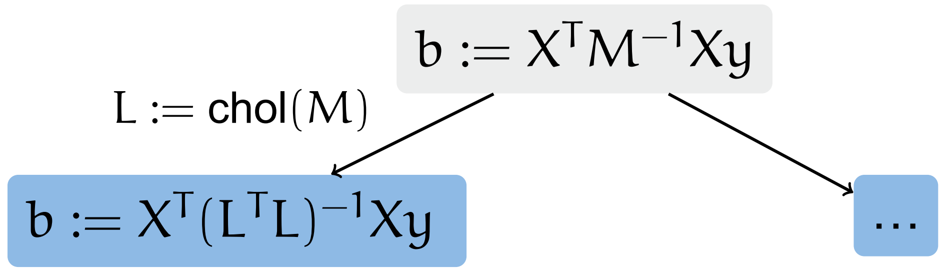
In practice:

$$X := AB^TC^{-T}D$$

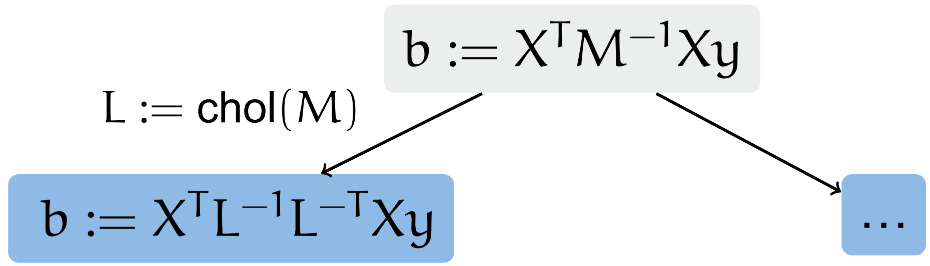
Derivation Graph

$$b := X^T M^{-1} X y$$

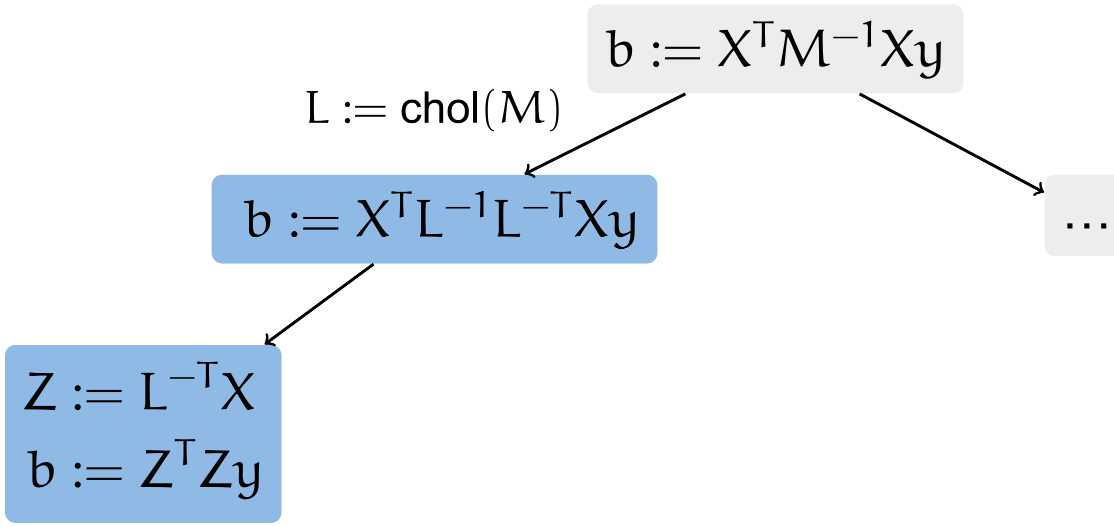
Derivation Graph



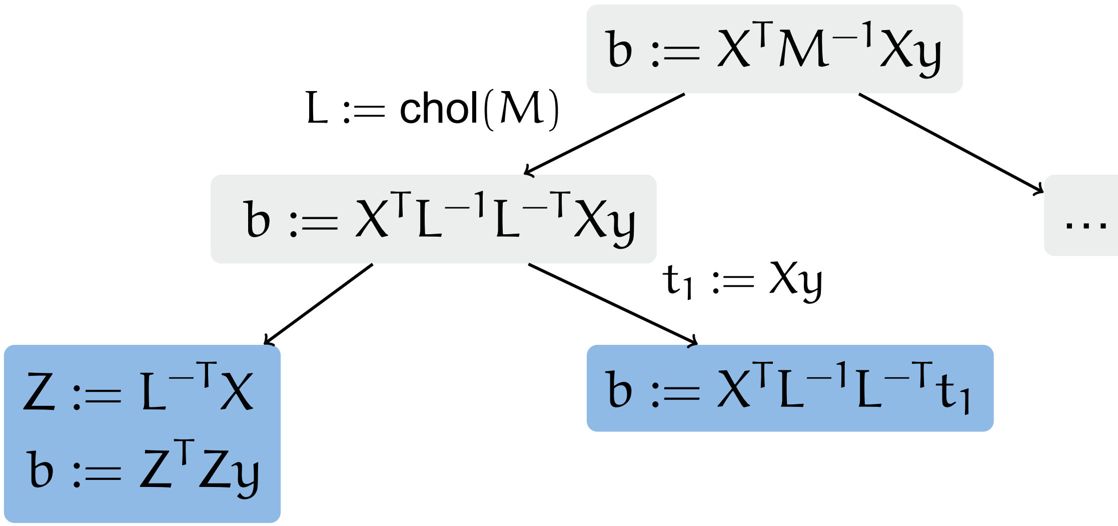
Derivation Graph



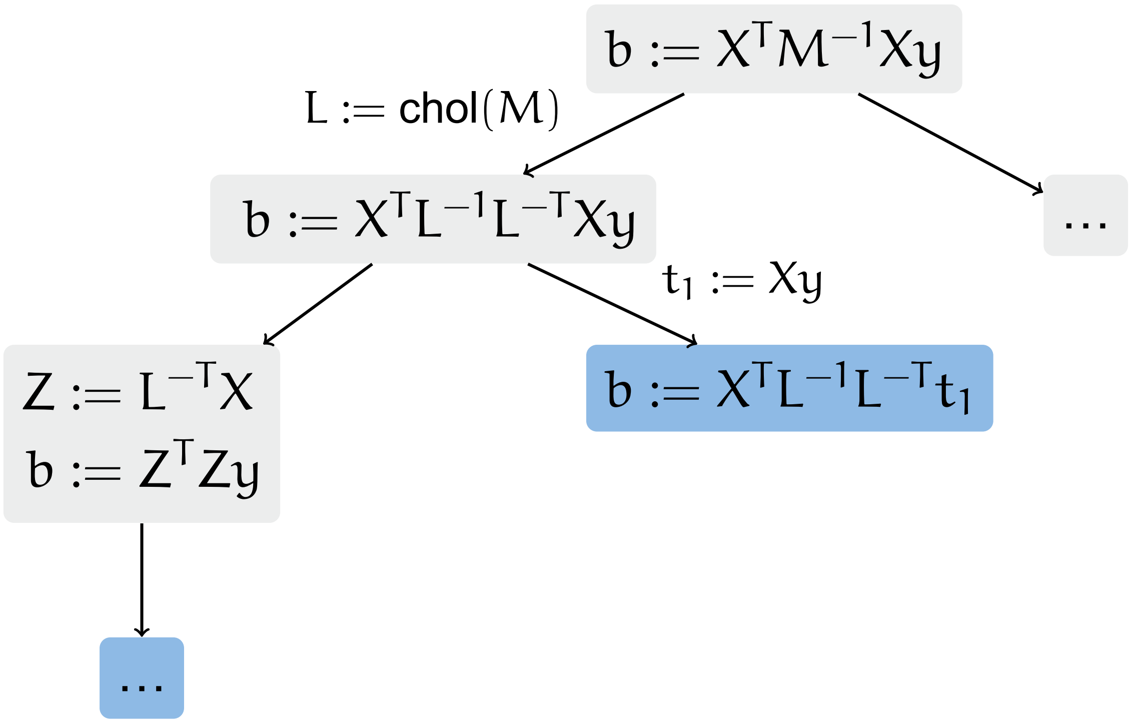
Derivation Graph



Derivation Graph

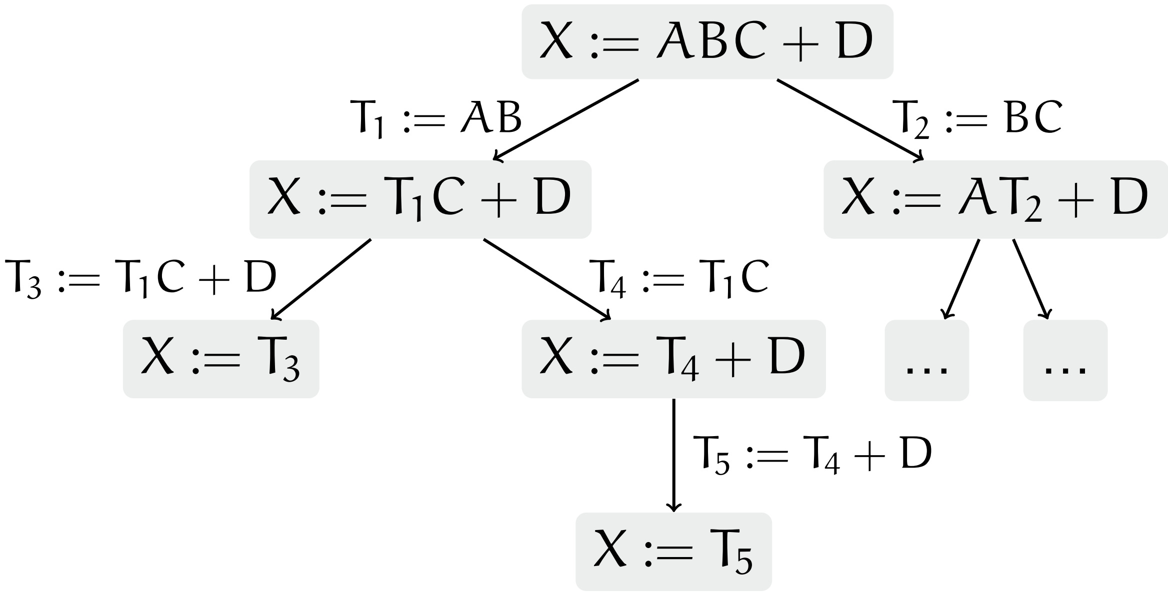


Derivation Graph

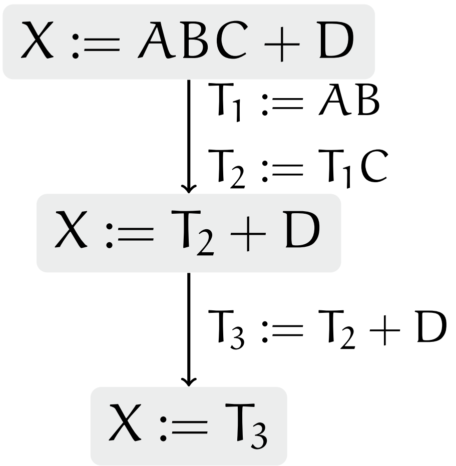


Derivation Graph

Exhaustive

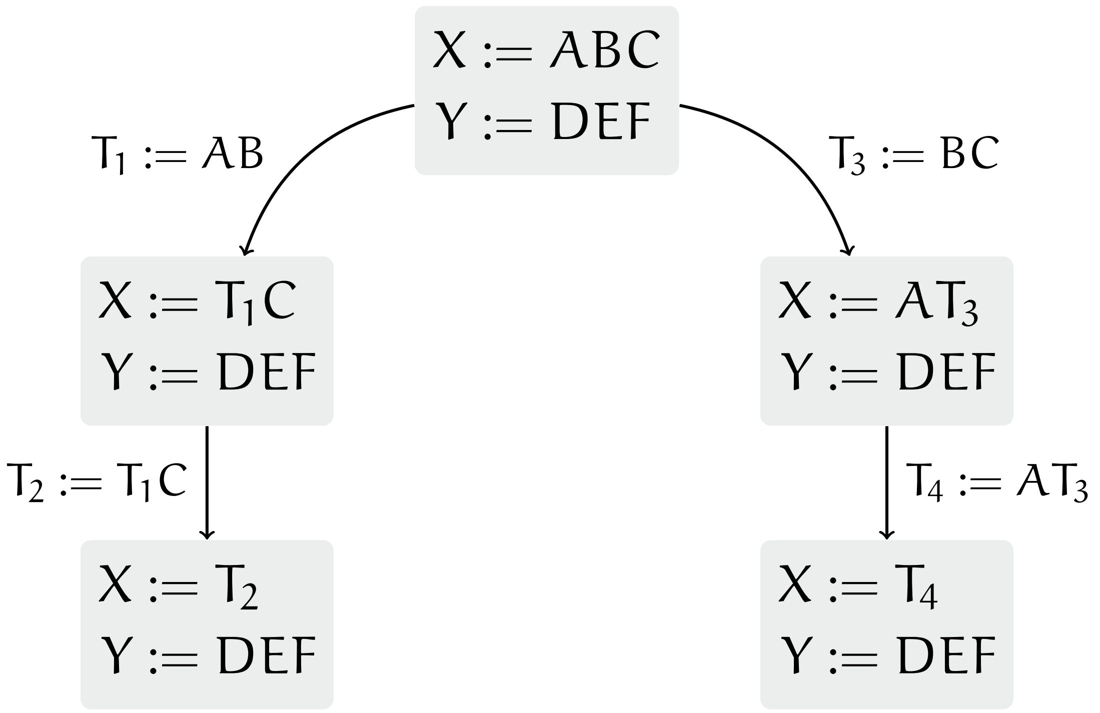


Constructive



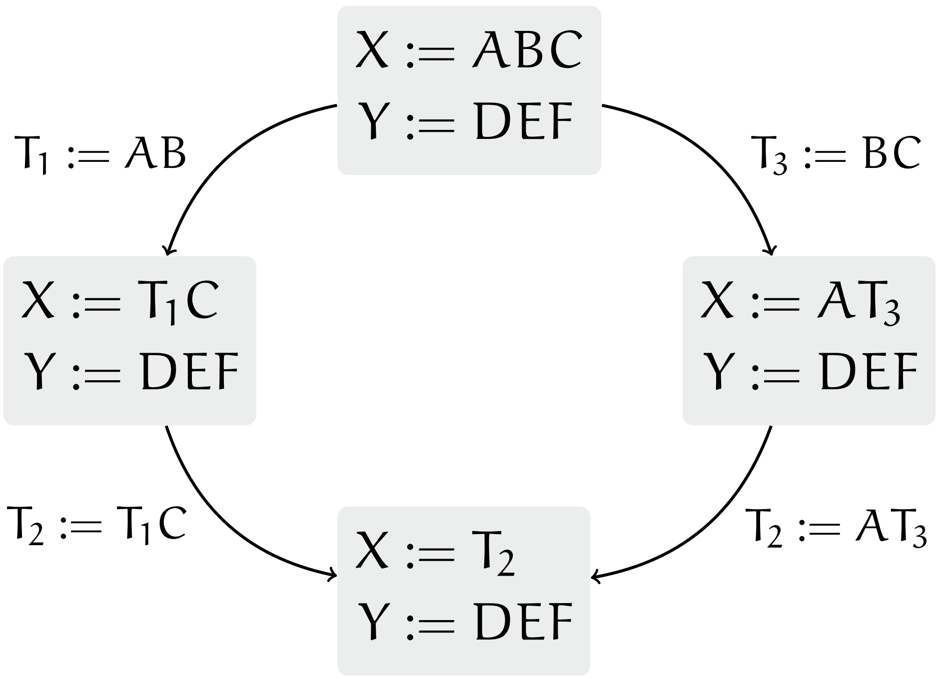
Derivation Graph

Reducing Redundancy



Derivation Graph

Reducing Redundancy



Results

Example: $w := AB^{-1}c$

Naive

```
w = A*inv(B)*c
```

Recommended

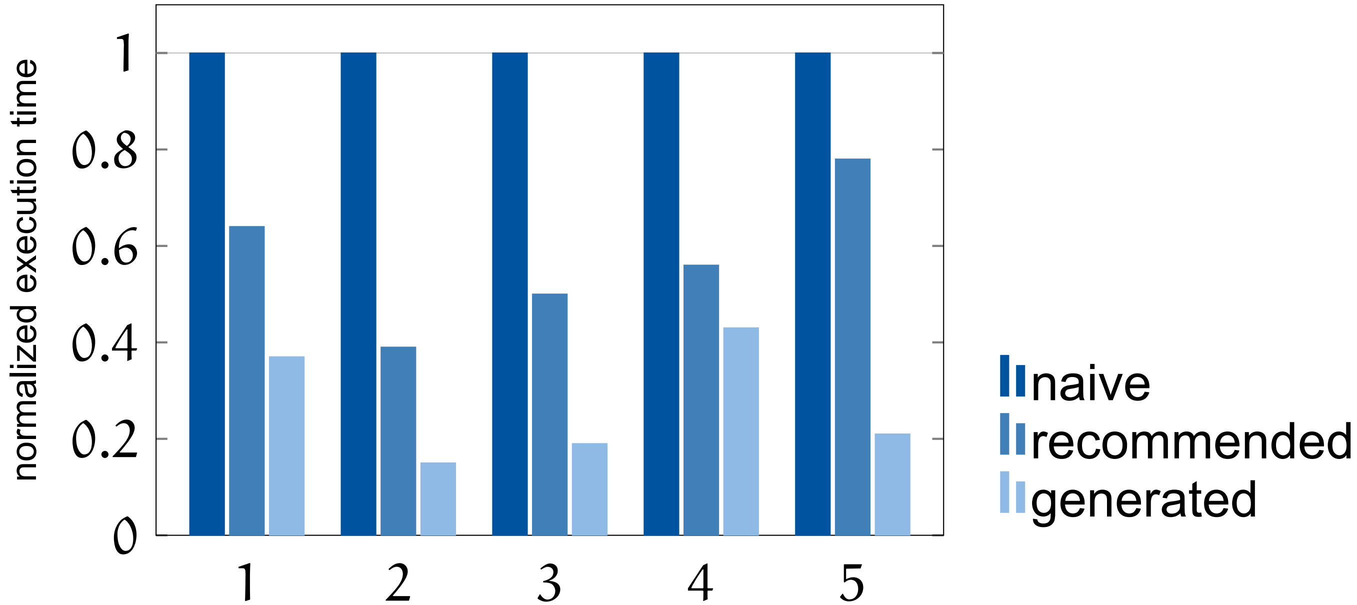
```
w = A*(B\c)
```

Generated

```
ml0 = A; ml1 = B; ml2 = c;
potrf!('L', ml1)
trsv!('L', 'N', 'N', ml1, ml2)
trsv!('L', 'T', 'N', ml1, ml2)
ml3 = Array{Float64}(10)
gemv!('N', 1.0, ml0, ml2, 0.0, ml3)
w = ml3
```

Results

#	Example	code gen time (ms)
1	$b := (X^T X)^{-1} X^T y$	37
2	$b := (X^T M^{-1} X)^{-1} X^T M^{-1} y$	430
3	$W := A^{-1} B C D^{-T} E F$	9
4	$X := A B^{-1} C;\ Y := D B^{-1} A^T$	17
5	$x := W(A^T (A W A^T)^{-1} b - c)$	537



References

[AB+99] Edward Anderson, Zhaojun Bai, et al. *LAPACK Users' guide*, volume 9. SIAM, 1999.

[DDC+90] Jack J. Dongarra, Jeremy Du Croz, et al. A set of Level 3 Basic Linear Algebra Subprograms. *ACM TOMS*, 16(1):1–17, 1990.