A Domain-Specific Compiler for Linear Algebra Operations

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7th Intl Workshop on Automatic Performance Tuning (iWAPT)
Kobe, July 17th, 2012

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... classic problems

- $b := \left(X^T X\right)^{-1} X^T y$

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- $\begin{cases} b_{ij} := (X_i^T M_j^{-1} X_i)^{-1} X_i^T M_j^{-1} y_j \\ M_j := h_j \Phi + (1 - h_j) I \end{cases}$

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  - ➥ Smart mapping onto BLAS/LAPACK
  - ➥ The decomposition is not unique: many algorithms

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➥ Smart mapping onto BLAS/LAPACK

➥ Different patterns require different algorithms

Linear Algebra Compiler

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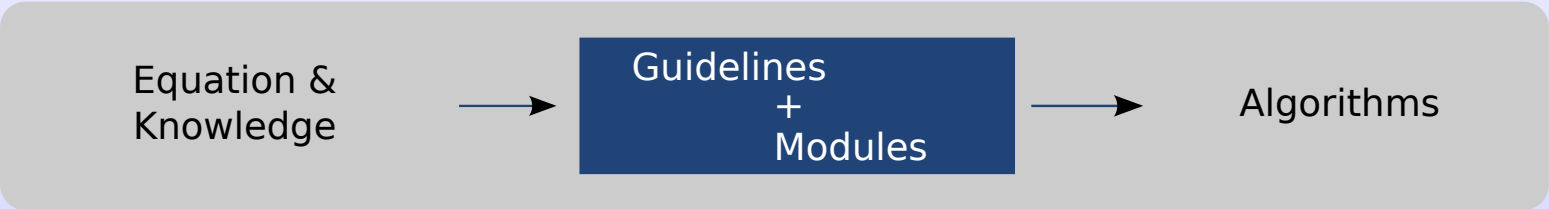
Heuristics: Search led by knowledge (mathematical and app-specific)

1. Automation
2. Automation (II)
3. What's next?
4. Conclusions

Automation

Automation



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  - Reducing flops: $S = R^{-1} Q^T Ly$

Modules

- Matrix Algebra
- Properties and Inference
- Kernels
- Sequences

Properties of the operands

- Size and shape
- Diagonal, Lower/Upper triangular
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- $A := X^T X \Rightarrow A$ is SPD

Encoded knowledge

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Database of building blocks

- BLAS / LAPACK
  - Factorizations: QR, LU, Cholesky, Eigen, ...
  - BLAS: GEMM, TRSM, GEMV, DOT, ...
  - LAPACK: inverse of a triangular matrix, ...
- Extensible

Example: GLS in GWAS

$$\begin{cases} b := (X^T M^{-1} X)^{-1} X^T M^{-1} y \\ M := h\Phi + (1-h)I \end{cases}$$

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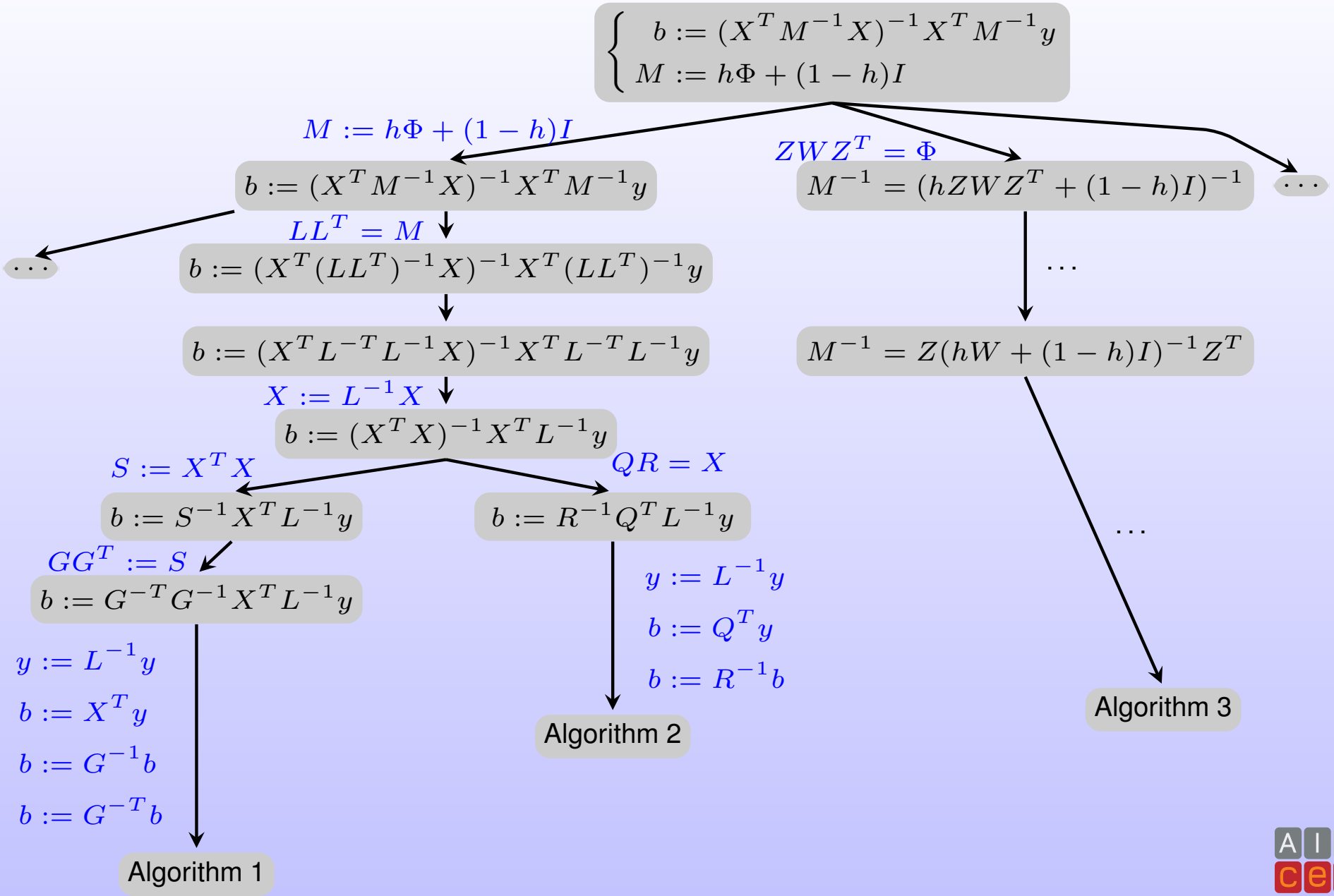

$$\begin{cases} b_{ij} = (X_i^T M_j^{-1} X_i)^{-1} X_i^T M_j^{-1} y_j & \text{with } 1 \leq i \leq m \\ M_j = h_j \Phi + (1 - h_j) I & \text{and } 1 \leq j \leq t. \end{cases}$$

- We have to solve not one but a sequence of **correlated** problems

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- We have to solve not one but a sequence of **correlated** problems
- Goal: **reuse of computation**

Naive approach: `for i, for j, ...`

$$b_{ij} = \left(X_i^T M_j^{-1} X_i\right)^{-1} X_i^T M_j^{-1} y_j$$

`for` $i = 1 : m$

  `for` $j = 1 : t$

    $LL^T = M_j$

    $X^T \leftarrow X_i^T L^{-T}$

    $QR = X$

    $y \leftarrow L^{-1} y_j$

    $b \leftarrow Q^T y$

    $b_{ij} \leftarrow R^{-1} b$

Tracking the dependencies

$$b_{ij} = \left(X_i^T M_j^{-1} X_i\right)^{-1} X_i^T M_j^{-1} y_j$$

$$\begin{aligned}
&\texttt{for } i = 1 : m \\
&\quad \texttt{for } j = 1 : t \\
&\qquad L_j L_j^T = M_j \\
&\qquad X_{ij}^T \leftarrow X_i^T L_j^{-T} \\
&\qquad Q_{ij} R_{ij} = X_{ij} \\
&\qquad y_j \leftarrow L_j^{-1} y_j \\
&\qquad b_{ij} \leftarrow Q_{ij}^T y_j \\
&\qquad b_{ij} \leftarrow R_{ij}^{-1} b_{ij}
\end{aligned}$$

Loop Transposition

$$b_{ij} = \left(X_i^T M_j^{-1} X_i\right)^{-1} X_i^T M_j^{-1} y_j$$

```
for i = 1 : m
    for j = 1 : t
        L_j L_j^T = M_j
        X_ij^T ← X_i^T L_j^{-T}
        Q_ij R_ij = X_ij
        y_j ← L_j^{-1} y_j
        b_ij ← Q_ij^T y_j
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for j = 1 : t
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Reordering

$$b_{ij} = \left(X_i^T M_j^{-1} X_i\right)^{-1} X_i^T M_j^{-1} y_j$$

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Steps

- Input: Dependencies operand $\leftrightarrow$ dimension
- Label each operand
- Propagation of dependencies
- Reordering
- Loop transposition

How to pick a variant?

- Many algorithms for a single target equation. How do we pick one?
- Flop count: Often not too descriptive

Scenario	Alg. 1	Alg. 2	Alg. 3
One instance	$O(n^3)$	$O(n^3)$	$O(n^3)$
2D sequence	$O(tn^3 + mtn^2)$	$O(tn^3 + mtn^2)$	$O(n^3 + mtn)$

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Next talk: Prediction

Conclusions and future work

So far...

- Domain-specific linear algebra compiler
- Guidelines + Modules + Knowledge
- Families of algorithms
- Not only a theoretical exercise:
  - AD: speedups ranging from 20 to 300
  - GWAS: speedups larger than 100

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TO-DO

- Encoding more available knowledge
- How to pick the “best” algorithm?
- Working on a Fortran code generator

Thanks to:

- Dr. Edoardo Di Napoli
- Matthias Petschow
- Roman Iakymchuk
- Elmar Peise

Financial support from the **Deutsche Forschungsgemeinschaft** (German Research Association) through grant GSC 111 is gratefully acknowledged.