Automation in matrix computations

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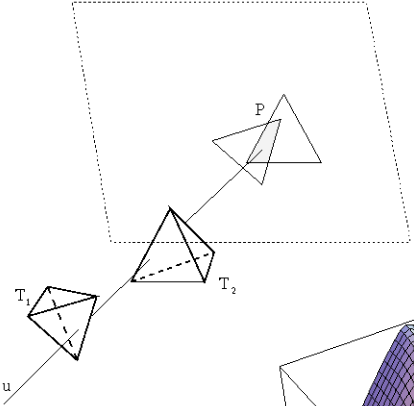
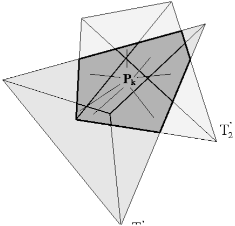
IFIP WG2.5, Valencia

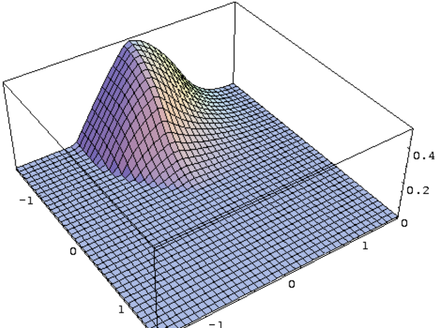
HPAC High Performance and Automatic Computing

Italy – Tuscany

Italy – University of Pisa







USA – UT Austin, TX

- Automatic generation of algorithms
- Symmetric eigenproblem $AX = X\Lambda$
- Parallel linear algebra
- Stability analysis

Algorithm $LU = A$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right), L \rightarrow \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right), U \rightarrow \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right)$

where $A_{TL}, L_{TL}, U_{TL},$ are 0×0

While $m(A_{TL}) \leq m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right),$$

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array}\right)$$

where $\alpha_{11}, 1, \upsilon_{11}$ are scalars

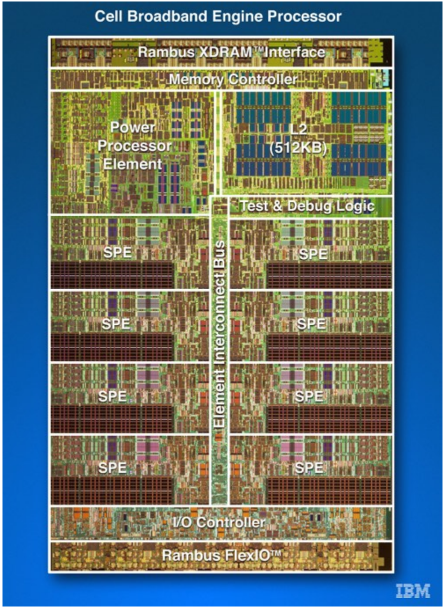
$$\upsilon_{11} := \alpha_{11} - l_{10}^T u_{01}$$
$$u_{12}^T := a_{12}^T - l_{10}^T U_{02}$$
$$l_{21} := (a_{21} - L_{20} u_{01}) / \upsilon_{11}$$

Continue with

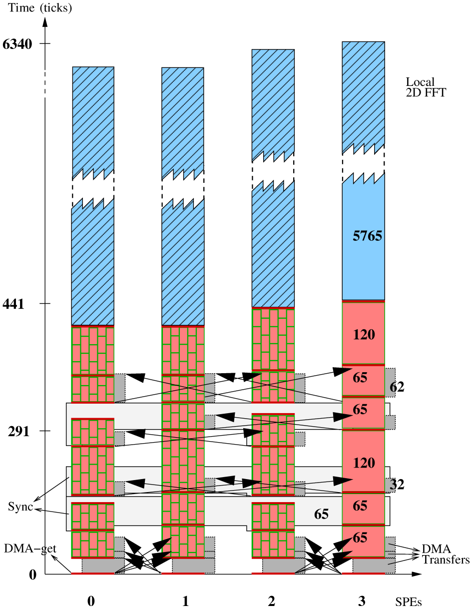
$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & 1 & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right), \cdots$$

endwhile

USA – Duke, NC



- Cell
- FFT



Germany – RWTH Aachen University

High-Performance &
Automatic Computing

IPCC
Molecular Dynamics

github.com/HPAC

Sweden – Umeå University

Tensor operations Linear Algebra compiler

The world of scientific computing

The world of scientific computing

$$V_{LJ} = 4\varepsilon \left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6} \right]$$

Lennard-Jones potential

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \left[\frac{-2\hbar}{2\mu} \nabla^2 + V(\mathbf{r}, t) \right] \Psi(\mathbf{r}, t)$$

Schrödinger eqn.

$$\boldsymbol{y} = X\boldsymbol{\beta} + Z\boldsymbol{u} + \boldsymbol{\epsilon}$$

$$\min_x \|A\mathbf{x} - \mathbf{b}\|^2 + \|\Gamma\mathbf{x}\|^2$$

Linear mixed models

$\vdots$

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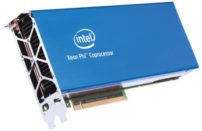

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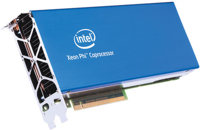

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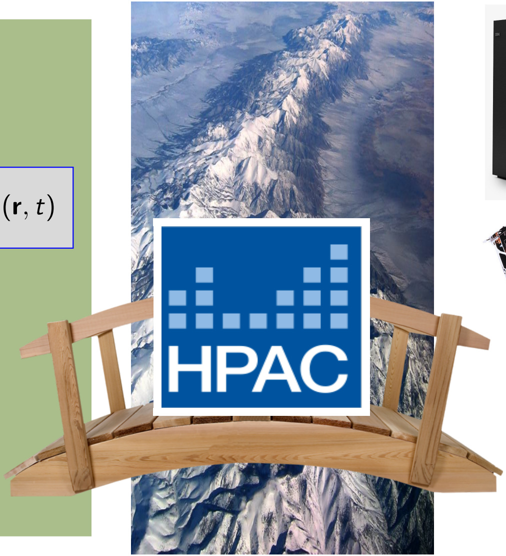
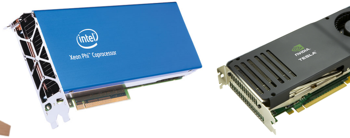
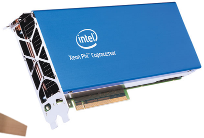




This talk

Matrix
computations

languages
libraries
compilers



Matrix computations

Signal Processing	$x := \left(A^{-T}B^TBA^{-1} + R^TLR\right)^{-1} A^{-T}B^TBA^{-1}y$	$R \in \mathbb{R}^{n-1\times n}$, UT; $L \in \mathbb{R}^{n-1\times n-1}$, DI
Kalman Filter	$K_k := P_k^bH^T(HP_k^bH^T + R)^{-1};\; x_k^a := x_k^b + K_k(z_k - Hx_k^b);\; P_k^a := (I - K_K H)\, P_k^b$	
Ensemble Kalman Filter	$X^a := X^b + \left(B^{-1} + H^TR^{-1}H\right)^{-1}\left(Y - HX^b\right)$	$B \in \mathbb{R}^{N\times N}$ SSPD; $R \in \mathbb{R}^{m\times m}$, SSPD
Ensemble Kalman Filter	$\delta X := \left(B^{-1} + H^TR^{-1}H\right)^{-1} H^TR^{-1}\left(Y - HX^b\right)$	
Ensemble Kalman Filter	$\delta X := XV^T\left(R + HX(HX)^T\right)^{-1}\left(Y - HX^b\right)$	
Image Restoration	$x_k := (H^TH + \lambda\sigma^2 I_n)^{-1}(H^Ty + \lambda\sigma^2(v_{k-1} - u_{k-1}))$	
Image Restoration	$H^\dagger := H^T(HH^T)^{-1};\; y_k := H^\dagger y + (I_n - H^\dagger H)x_k$	
Rand. Matrix Inversion	$X_{k+1} := S(S^TAS)^{-1}S^T + (I_n - S(S^TAS)^{-1}S^TA)X_k(I_n - AS(S^TAS)^{-1}S^T)$	
Rand. Matrix Inversion	$X_{k+1} := X_k + WA^TS(S^TAWA^TS)^{-1}S^T(I_n - AX_k)$	$W \in \mathbb{R}^{n\times n}$, SPD
Rand. Matrix Inversion	$X_{k+1} := X_k + (I_n - X_kA^T)S(S^TA^TWAS)^{-1}S^TA^TW$	
Rand. Matrix Inversion	$\Lambda := S(S^TAWAS)^{-1}S^T;\; \Theta := \Lambda AW;\; M_k := X_kA - I$ $X_{k+1} := X_k - M_k\Theta - (M_k\Theta)^T + \Theta^T(AX_kA - A)\Theta$	

Matrix computations (2)

Generalized Least Squares	$b := (X^T M^{-1} X)^{-1} X^T M^{-1} y$	$n > m$; $M \in \mathbb{R}^{n \times n}$, SPD; $X \in \mathbb{R}^{n \times m}$; $y \in \mathbb{R}^{n \times 1}$
Stochastic Newton	$B_k := \frac{k}{k-1} B_{k-1}(I_n - A^T W_k((k-1)I_l + W_k^T A B_{k-1} A^T W_k)^{-1} W_k^T A B_{k-1})$	
Optimization	$x_f := W A^T (A W A^T)^{-1}(b - Ax)$; $x_o := W(A^T (A W A^T)^{-1} A x - c)$	
Optimization	$x := W(A^T (A W A^T)^{-1} b - c)$	
Triangular Matrix Inv.	$X_{10} := L_{10} L_{00}^{-1}$; $X_{20} := L_{20} + L_{22}^{-1} L_{21} L_{11}^{-1} L_{10}$; $X_{11} := L_{11}^{-1}$; $X_{21} := -L_{22}^{-1} L_{21}$	
Tikhonov Regularization	$x := (A^T A + \Gamma^T \Gamma)^{-1} A^T b$	$A \in \mathbb{R}^{n \times m}$; $\Gamma \in \mathbb{R}^{m \times m}$; $b \in \mathbb{R}^{n \times 1}$
Tikhonov Regularization	$x := (A^T A + \alpha^2 I)^{-1} A^T b$	
Gen. Tikhonov Reg.	$x := (A^T P A + Q)^{-1}(A^T P b + Q x_0)$	$P \in \mathbb{R}^{n \times n}$, SSPD; $Q \in \mathbb{R}^{m \times m}$, SSPD; $x_0 \in \mathbb{R}^{m \times 1}$
Gen. Tikhonov reg.	$x := x_0 + (A^T P A + Q)^{-1}(A^T P(b - A x_0))$	
LMMSE estimator	$K_{t+1} := C_t A^T (A C_t A^T + C_z)^{-1}$; $x_{t+1} := x_t + K_{t+1}(y - A x_t)$; $C_{t+1} := (I - K_{t+1} A) C_t$	
LMMSE estimator	$x_{\text{out}} = C_X A^T (A C_X A^T + C_Z)^{-1}(y - Ax) + x$	
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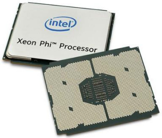

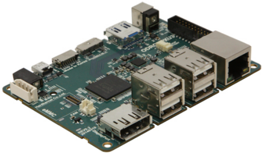

MUL ADD MOV

MOVAPD

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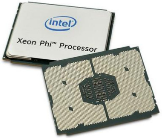

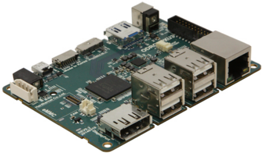

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2-step solution

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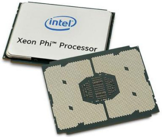

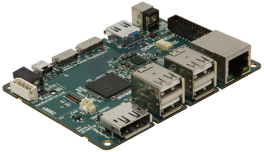

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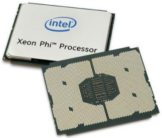

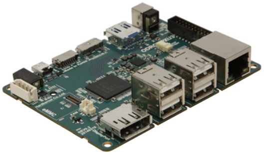

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LINEAR ALGEBRA MAPPING PROBLEM (LAMP)

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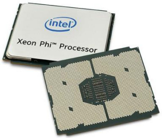

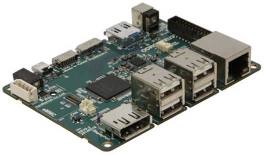

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- Suboptimal solution easy
- Optimality NP complete reduction from Ensemble Computation

Problem acknowledged, yet overlooked

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Slow solutions

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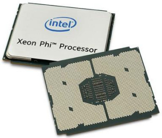
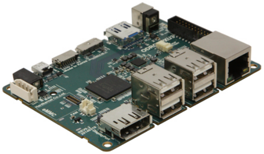
  Slow solutions $\rightarrow$ "2-language problem"

Quantum Chemistry | High-order FEM | DG | ... | Tensor App #N

?





MUL ADD MOV

MOVAPD

VFMADDPD ...

Quantum Chemistry | High-order FEM | DG | ... | Tensor App #N

? ⬇ ?

Transposition | Contraction | ⋯ | Alternating LS

Khatri-Rao | SpMTTKRP | ⋯ | TTV, TTM

HPTT ⬇ **TCL** ⬇ **...** ⬇ **BLAS**

MUL | ADD | MOV

MOVAPD

VFMADDPD ...

Matrix computations: State of the art

High level of abstraction: Matlab, Julia, R, Eigen, Armadillo, NumPy

Why? Popularity, expressiveness, (performance...)

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Why? Popularity, expressiveness, (performance...)

```
C = A * B'+ B * A' + C;                           // Matlab

C = A * transpose(B) + B * transpose(A) + C       // Julia

C = A * trans(B) + B * trans(A) + C;              // Armadillo

C = A * B.transpose() + B * A.transpose() + C;    // Eigen

ct = at @ bt.T + bt @ at.T + ct                   // NumPy

ct <- at %*% t(bt) + bt %*% t(at) + ct            // R
```

Do they map? matrix products

	Matlab	Julia	R	Eigen	Armad.	NumPy	C
C = AB+C							

Do they map? matrix products

	Matlab	Julia	R	Eigen	Armad.	NumPy	C
$C = AB + C$	0.28	0.31	0.30	0.29	0.28	0.29	0.26

Do they map? matrix products

	Matlab	Julia	R	Eigen	Armad.	NumPy	C
C $=$ AB$+$C	0.28	0.31	0.30	0.29	0.28	0.29	0.26
C $=$ AB	...						
C $=$ αAB							
C $=$ αAB$+\beta$C							

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C $=$ AB$+$C	0.28	0.31	0.30	0.29	0.28	0.29	0.26
C $=$ AB	. . .						
C $=$ αAB							
C $=$ αAB$+\beta$C							
`GEMM`	✓	✓	✓	✓	✓	✓	

Do they map? matrix products

	Matlab	Julia	R	Eigen	Armad.	NumPy	C
C = AB+C	0.28	0.31	0.30	0.29	0.28	0.29	0.26
C = AB	. . .						
C = αAB							
C = αAB+βC							
`GEMM`	✓	✓	✓	✓	✓	✓	
C = C+AA'	0.17	0.22	0.31	0.29	0.17	0.18	0.13

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C = AB+C	0.28	0.31	0.30	0.29	0.28	0.29	0.26
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`SYRK`	✓	✓	×	×	✓	✓	

Do they map? matrix products

	Matlab	Julia	R	Eigen	Armad.	NumPy	C
C = AB+C	0.28	0.31	0.30	0.29	0.28	0.29	0.26
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`GEMM`	✓	✓	✓	✓	✓	✓	
C = C+AA'	0.17	0.22	0.31	0.29	0.17	0.18	0.13
`SYRK`	✓	✓	×	×	✓	✓	
C = C+AB'+BA'	0.56	0.70	0.61	0.57	0.56	0.58	0.27

Do they map? matrix products

	Matlab	Julia	R	Eigen	Armad.	NumPy	C
C = AB+C	0.28	0.31	0.30	0.29	0.28	0.29	0.26
C = AB	...						
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C = αAB+βC							
`GEMM`	✓	✓	✓	✓	✓	✓	
C = C+AA'	0.17	0.22	0.31	0.29	0.17	0.18	0.13
`SYRK`	✓	✓	×	×	✓	✓	
C = C+AB'+BA'	0.56	0.70	0.61	0.57	0.56	0.58	0.27
`SYR2K`	×	×	×	×	×	×	

Do they map? $Ax = b \quad \equiv \quad x := A\backslash b \quad \not\equiv \quad \mathrm{inv}(A) * b$

	Matlab	Julia	R	Eigen	Armad.	NumPy	C
$x := A\backslash b$							

Do they map? $Ax = b \quad \equiv \quad x := A \backslash b \quad \not\equiv \quad \mathrm{inv}(A) * b$

	Matlab	Julia	R	Eigen	Armad.	NumPy	C
$x := A \backslash b$	0.70	0.62	0.67	0.63	0.62	0.65	0.61

Do they map? $Ax = b \quad \equiv \quad x := A\backslash b \quad \not\equiv \quad \text{inv}(A) * b$

	Matlab	Julia	R	Eigen	Armad.	NumPy	C
$x := A\backslash b$	0.70	0.62	0.67	0.63	0.62	0.65	0.61
$\text{inv}(A) * b$	1.74	1.45	2.20	2.20	**0.62**	2.22	1.71

Do they map? $Ax = b \quad \equiv \quad x := A\backslash b \quad \not\equiv \quad \text{inv}(A) * b$

	Matlab	Julia	R	Eigen	Armad.	NumPy	C
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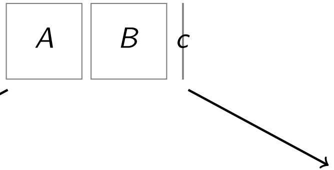
$\text{inv}(A) * b$	1.74	1.45	2.20	2.20	**0.62**	2.22	1.71
LinSolve	-	-	-	-	✓	-	-

... should they map?

Parenthesisation



$(AB)c \quad \mathcal{O}(n^3)$

$A(Bc) \quad \mathcal{O}(n^2)$

Product is associative, but its cost is not

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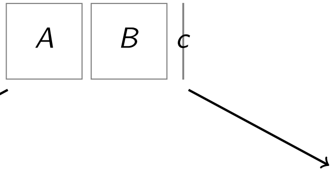

$(AB)c \quad \mathcal{O}(n^3)$

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Product is associative, but its cost is not

Matrix Chain Algorithm $O(k \log k)$ Hu & Shing 1982; $O(k^3)$ dynamic programming

Matrix Chain?

Chain	Optimal Evaluation
1) “left-to-right” (LtR)	((A B) C)
2) “right-to-left” (RtL)	(A (B C))
3) “mixed” (Mix)	((A B) (C D))

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3) "mixed" (Mix)	((A B) (C D))

		Matlab	Julia	R	Eigen	Armad.	NumPy
1)	LtR no par.	0.056	0.055	0.061	0.058	0.056	0.055
	LtR guided	0.056	0.055	0.061	0.058	0.056	0.055

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1) "left-to-right" (LtR)	((A B) C)
2) "right-to-left" (RtL)	(A (B C))
3) "mixed" (Mix)	((A B) (C D))

		Matlab	Julia	R	Eigen	Armad.	NumPy
1)	LtR no par.	0.056	0.055	0.061	0.058	0.056	0.055
	LtR guided	0.056	0.055	0.061	0.058	0.056	0.055
2)	RtL no par.	0.42	0.42	0.44	0.42	**0.055**	0.42
	RtL guided	0.055	0.054	0.059	0.056	0.055	0.056

Matrix Chain?

Chain	**Optimal Evaluation**
1) "left-to-right" (LtR)	((A B) C)
2) "right-to-left" (RtL)	(A (B C))
3) "mixed" (Mix)	((A B) (C D))

		Matlab	Julia	R	Eigen	Armad.	NumPy
1)	LtR no par.	0.056	0.055	0.061	0.058	0.056	0.055
	LtR guided	0.056	0.055	0.061	0.058	0.056	0.055
2)	RtL no par.	0.42	0.42	0.44	0.42	**0.055**	0.42
	RtL guided	0.055	0.054	0.059	0.056	0.055	0.056
3)	Mix no par.	0.32	0.33	0.33	0.35	0.31	0.33
	Mix guided	0.21	0.22	0.22	0.23	0.20	0.22

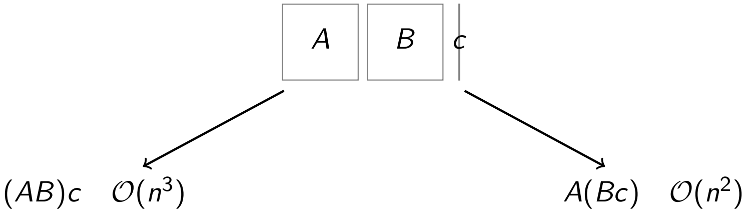
Matrix Chain?

Chain	Optimal Evaluation
1) "left-to-right" (LtR)	((A B) C)
2) "right-to-left" (RtL)	(A (B C))
3) "mixed" (Mix)	((A B) (C D))

		Matlab	Julia	R	Eigen	Armad.	NumPy
1)	LtR no par.	0.056	0.055	0.061	0.058	0.056	0.055
	LtR guided	0.056	0.055	0.061	0.058	0.056	0.055
2)	RtL no par.	0.42	0.42	0.44	0.42	**0.055**	0.42
	RtL guided	0.055	0.054	0.059	0.056	0.055	0.056
3)	Mix no par.	0.32	0.33	0.33	0.35	0.31	0.33
	Mix guided	0.21	0.22	0.22	0.23	0.20	0.22
	Matrix chains	$\times$	$\times$	$\times$	$\times$	$\approx$	$\times$

Challenge: Generalized Matrix Chain

[arXiv:1804.04021]



In practice

- Unary operators: transposition, inversion $(X := AB^T C^{-T} D + \dots)$
- Overlapping kernels (e.g., $L \leftarrow L^{-1}$, $X = A^{-1}B$)
- Decompositions (e.g., $A \to Q^T DQ$, $A \to LU$)
- Properties & specialized kernels (GEMM, TRMM, SYMM, ...)

Challenge: Not all flops were created equal

$\#$FLOPs vs. execution time (vs. numerical stability)

$$\underset{\mathcal{A}}{\text{argmin}}\,(\ \text{FLOPs}(\mathcal{A})\ ) \neq \underset{\mathcal{A}}{\text{argmin}}\,(\ \text{time}(\mathcal{A})\ )$$

Challenge: Not all flops were created equal [arXiv:1706.01341]

#FLOPs vs. execution time (vs. numerical stability)

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$$\operatorname*{argmin}_{\mathcal{A}} (\ \text{FLOPs}(\mathcal{A})\ ) \neq \operatorname*{argmin}_{\mathcal{A}} (\ \text{time}(\mathcal{A})\ )$$

$\Rightarrow$ **Performance prediction**: efficiency

Parallelism: $X := A((B^T C^{-T})D)$ vs. $X := (AB^T)(C^{-T}D)$ vs. ...

Properties?

Operation	Property	Matlab	Julia	R	Eigen	Armad.	NumPy	C
Linear System	-	0.70	0.62	0.67	0.63	0.62	0.65	0.61

Properties?

Operation	Property	Matlab	Julia	R	Eigen	Armad.	NumPy	C
Linear System	-	0.70	0.62	0.67	0.63	0.62	0.65	0.61
	Symmetric	0.71	0.62	0.69	N/A	0.62	0.65	0.46

Properties?

Operation	Property	Matlab	Julia	R	Eigen	Armad.	NumPy	C
Linear System	-	0.70	0.62	0.67	0.63	0.62	0.65	0.61
	Symmetric	0.71	0.62	0.69	N/A	0.62	0.65	0.46
	SPD	**0.41**	0.60	0.63	N/A	**0.34**	0.62	0.31

Properties?

Operation	Property	Matlab	Julia	R	Eigen	Armad.	NumPy	C
Linear System	-	0.70	0.62	0.67	0.63	0.62	0.65	0.61
	Symmetric	0.71	0.62	0.69	N/A	0.62	0.65	0.46
	SPD	**0.41**	0.60	0.63	N/A	**0.34**	0.62	0.31
	Triangular	**0.03**	**0.03**	0.63	N/A	0.62	0.65	0.03

Properties?

Operation	Property	Matlab	Julia	R	Eigen	Armad.	NumPy	C
Linear System	-	0.70	0.62	0.67	0.63	0.62	0.65	0.61
	Symmetric	0.71	0.62	0.69	N/A	0.62	0.65	0.46
	SPD	**0.41**	0.60	0.63	N/A	**0.34**	0.62	0.31
	Triangular	**0.03**	**0.03**	0.63	N/A	0.62	0.65	0.03
	Diagonal	0.03	**0.01**	0.63	N/A	**0.03**	0.62	0.001
		$\approx$	$\approx$	$\times$	$\times$	$\approx$	$\times$	

Properties?

Operation	Property	Matlab	Julia	R	Eigen	Armad.	NumPy	C
Linear System	-	0.70	0.62	0.67	0.63	0.62	0.65	0.61
	Symmetric	0.71	0.62	0.69	N/A	0.62	0.65	0.46
	SPD	**0.41**	0.60	0.63	N/A	**0.34**	0.62	0.31
	Triangular	**0.03**	**0.03**	0.63	N/A	0.62	0.65	0.03
	Diagonal	0.03	**0.01**	0.63	N/A	**0.03**	0.62	0.001
		$\approx$	$\approx$	$\times$	$\times$	$\approx$	$\times$	
Multiplication	-	1.44	1.47	1.47	1.45	1.44	1.44	1.46
	Triangular	1.44	**0.75**	1.47	1.45	1.44	1.44	0.74
	Diagonal	1.44	**0.03**	1.47	1.45	1.42	1.44	0.06
		$\times$	$\checkmark$	$\times$	$\times$	$\times$	$\times$	

Challenge: Inference of properties

- **easy** $E := L_1 * U^T * L_2$ triangular(E) ?

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Challenge: Inference of properties

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- **impossible?** $E := Q^{-1} U (I + U^T Q^{-1} U)^{-1} U^T$ properties($I + U^T Q^{-1} U$) ?

$\Rightarrow$ **Symbolic analysis**: pattern matching

Challenge: Common Subexpressions

[arXiv:1710.06915]

$$\begin{cases} X := AB^{-T}C \\ Y := B^{-1}A^{T}D \end{cases} \quad \rightarrow \quad \begin{cases} Z := AB^{-T} \\ X := ZC \\ Y := Z^{T}D \end{cases}$$

Challenge: Common Subexpressions [arXiv:1710.06915]

$$\begin{cases} X := AB^{-T}C \\ Y := B^{-1}A^{T}D \end{cases} \quad \to \quad \begin{cases} Z := AB^{-T} \\ X := ZC \\ Y := Z^{T}D \end{cases}$$

BUT

$$X := ABABv \quad \not\to \quad \begin{cases} Z := AB \\ X := ZZv \end{cases}$$

Common Subexpressions?

$$\begin{cases} X := AB \\ Y := AB \end{cases} \quad \to \quad \begin{cases} X := AB \\ Y := X \end{cases}$$

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$$\begin{cases} X := AB \\ Y := AB \end{cases} \rightarrow \begin{cases} X := AB \\ Y := X \end{cases}$$

	Matlab	Julia	R	Eigen	Armad.	NumPy
direct	0.54	0.61	0.56	0.58	0.52	0.55

Common Subexpressions?

$$\begin{cases} X := AB \\ Y := AB \end{cases} \rightarrow \begin{cases} X := AB \\ Y := X \end{cases}$$

	Matlab	Julia	R	Eigen	Armad.	NumPy
direct	0.54	0.61	0.56	0.58	0.52	0.55
copy	0.27	0.36	0.30	0.30	0.26	0.30
	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

Other features (challenges)

- Code motion

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for i = 1:n,
    X = A*B;
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$\times$

- Blocked operands

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

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- $\text{diag}(A + B)$ vs. $\text{diag}(A) + \text{diag}(B)$

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- diag$(A + B)$ vs. diag(A) + diag(B) $\rightarrow$ Armadillo

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- $\text{diag}(A + B)$ vs. $\text{diag}(A) + \text{diag}(B)$ $\rightarrow$ Armadillo

  $\text{diag}(AB)$ vs. ...

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- diag$(A + B)$ vs. diag(A) + diag(B) $\rightarrow$ Armadillo

  diag(AB) vs. ... $\rightarrow$ $\times$

Summary

- LAMP is challenging — lots of expertise needed; interdisciplinary

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- **Linnea**: A linear algebra compiler

Linear algebra knowledge: operators, identities, properties, theorems

- Distributivity, commutativity, partitionings, ...
- $((QR)^T QR)^{-1}(QR)^T y \rightarrow (R^T Q^T QR)^{-1} R^T Q^T y \rightarrow R^{-1} R^{-T} R^T Q^T y \rightarrow R^{-1} Q^T y$
- $\text{SPD}(A) \rightarrow \text{SPD}(A_{BR} - A_{BL} A_{TL}^{-1} A_{BL}^T)$ Schur complement
- $\cdots$

Linnea: Example

$$w := AB^{-1}c, \quad \mathsf{SPD}(B)$$

Naive

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w = A*inv(B)*c
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Recommended

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w = A*(B\c)
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Expert

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L = Chol(B)
w = A*(L'\(L\c))
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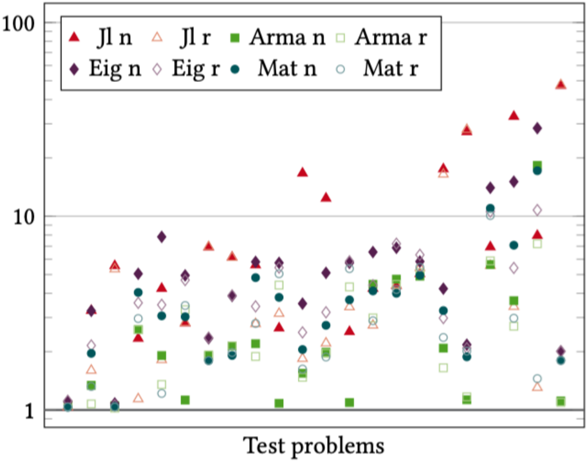
Generated

```
ml0 = A; ml1 = B; ml2 = c;
potrf!('L', ml1)
trsv!('L', 'N', 'N', ml1, ml2)
trsv!('L', 'T', 'N', ml1, ml2)
ml3 = Array{Float64}(10)
gemv!('N', 1.0, ml0, ml2, 0.0, ml3)
w = ml3
```

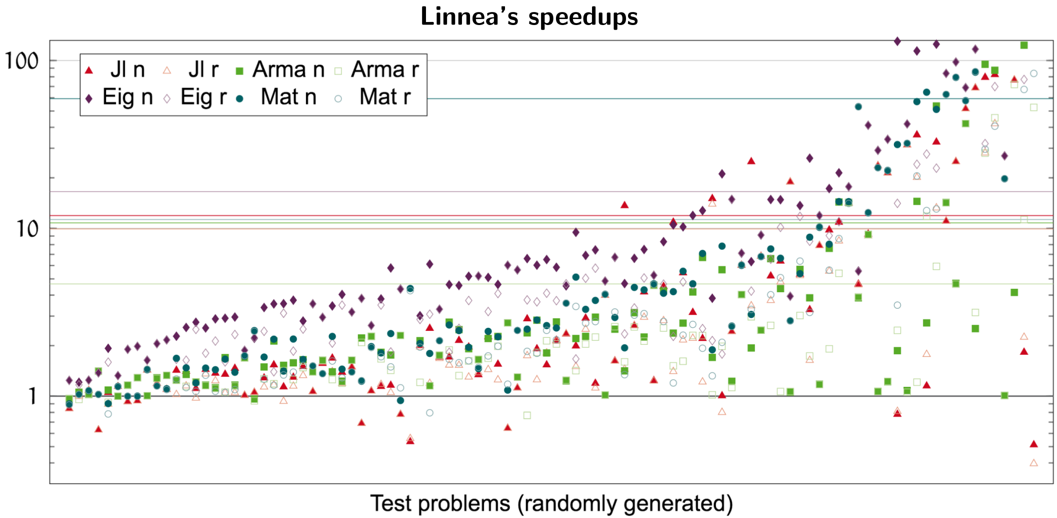
Results — applications

Linnea's speedups



Jl: Julia, **Arma**: Armadillo, **Eig**: Eigen, **Mat**: Matlab. **n**/**r**: naive/recommended implementation

Results — random expressions



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