High-performance Matrix Computations

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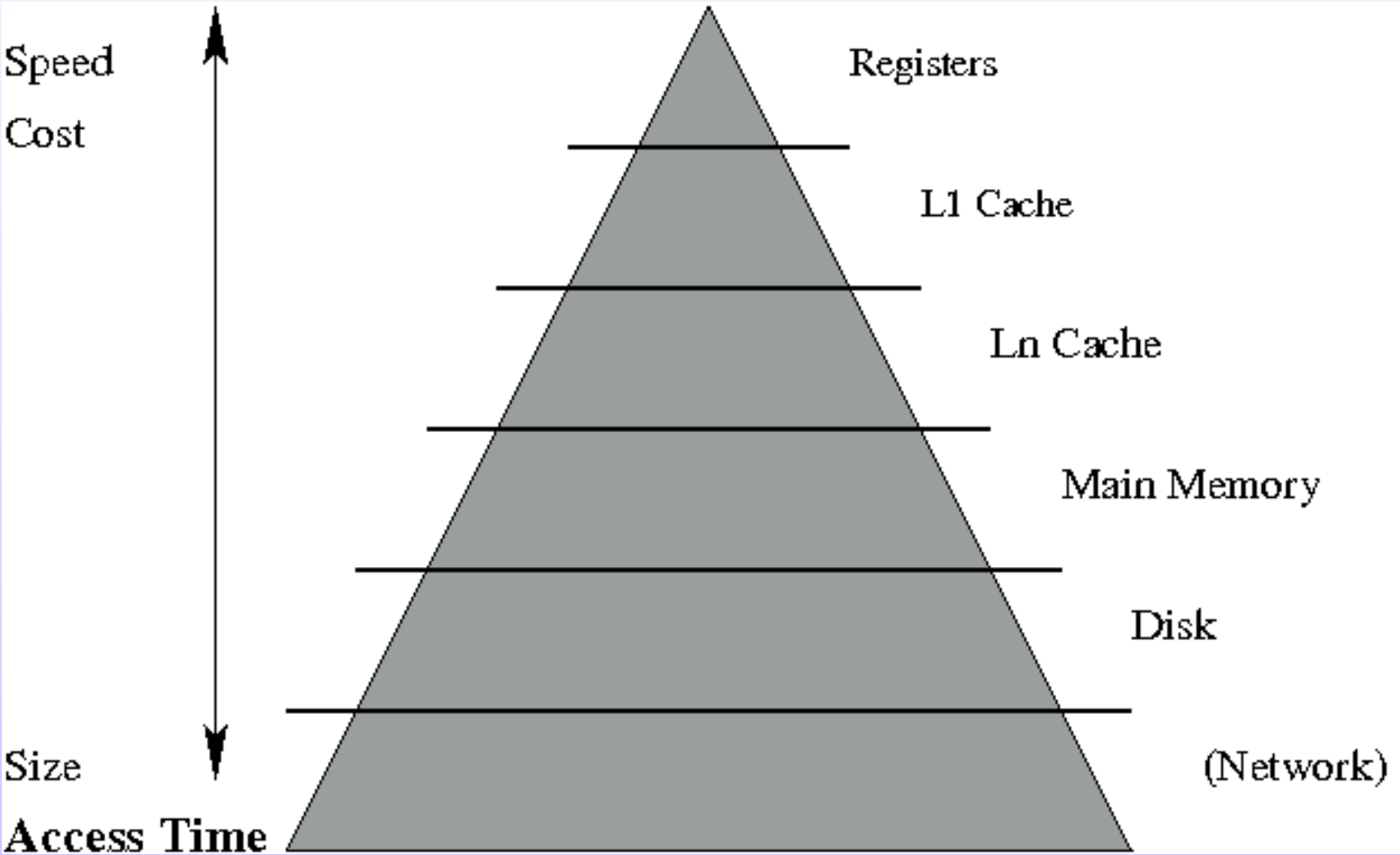
Memory Hierarchy



CPU vs. Memory

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- Cache misses
- Cache line
- L1 misses, L2, L3, TLB, instruction misses. . .
- Prefetching (HW, SW)

Locality

"Locality of references", "Principle of locality"

- Temporal locality
- Spatial locality

Performance

Quality of an algorithm

Metrics?

Performance

Quality of an algorithm

Metrics?

- Execution time
- Complexity
- Stability
- Accuracy
- Memory usage
- Memory/network accesses
- ...

Execution time

Pros? Cons?

Execution time

Pros? Cons?

- It says how fast an algorithm is
- “Does not measure the memory usage”
- “It might depend on the input”
- “It depends on the processor”
- **It does not measure how well the algorithm takes advantage of a given architecture**

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TPP is unattainable. Why?

Back to performance

- Theroretical peak performance $\rightarrow$ unattainable
- Practical peak performance? (practical = attainable)
- DGEMM (BLAS)
  Double GEneral Matrix Matrix multiply
  Basic Linear Algebra Subroutines
- Peak performance $\equiv$ DGEMM

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- How to compute DGEMM's performance?

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$$\text{Efficiency} = \frac{\text{Perf}}{\text{Peak Perf}}$$

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USE BOTH!

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"GEMM is holding back algorithmic innovation for tensor computations."
J.H.

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