HOUSEHOLDER REFLECTORS

$$H(v) = I - \beta v v^T, \quad \beta = \frac{2}{\|v\|_2^2} \qquad H = \underbrace{\begin{bmatrix}1 & & \\ & \ddots & \\ & & 1\end{bmatrix}}_{I} - \beta \underbrace{\Big|}_{v}\ \underbrace{\overline{\qquad}}_{v^T}$$

- EXERCISES: $H^T = H$, $HH^T = H^TH = H^2 = I$. H is UNITARY

$$\boxed{\text{For a given } x \in \mathbb{R}^m, \ \exists v \in \mathbb{R}^m. \ H(v)\cdot x = \alpha e_1} \qquad e_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} \Rightarrow x = \begin{bmatrix}x\\x\\x\\x\\x\end{bmatrix},\ Hx = \begin{bmatrix}*\\0\\\vdots\\0\end{bmatrix}$$

$(x \neq 0)$

$|\alpha| = \|x\|_2 \Rightarrow \alpha = \pm\|x\|_2$

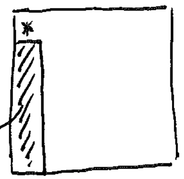
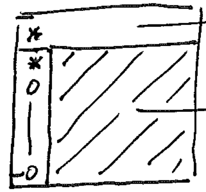
$v = x - \alpha e_1$ → THE SIGN OF α IS CHOSEN SO THAT $x - \alpha e_1$ IS AN ADDITION

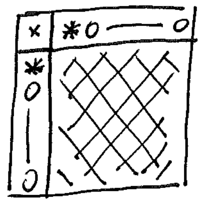
- EXERCISE: $\|x\|_2 = \|Hx\|_2 = |\alpha|$

- GIVEN $x \in \mathbb{R}^m = \begin{bmatrix}x\\x\\x\\x\\x\\x\\x\\x\end{bmatrix}$, WE WANT $\begin{array}{l}\begin{bmatrix}x\\x\\x\\x\\ * \\0\\\vdots\\0\end{bmatrix}\end{array}\Big\}k$ $\Rightarrow x = \begin{bmatrix}x_T\\ \hline x_B\end{bmatrix}\begin{array}{l}\}m-k\\ \}k\end{array}$, $\exists v \in \mathbb{R}^k$. $H^{k\times k}(v)\cdot x_B = \begin{bmatrix}*\\0\\\vdots\\0\end{bmatrix}$

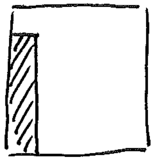
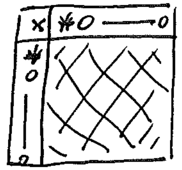
$$\Rightarrow Q^{m\times m} = \left[\begin{array}{c|c} I^{m-k} & \\ \hline & H(v)\end{array}\right] \text{ and } Qx = \begin{bmatrix}x_T\\ \hline *\\0\\\vdots\\0\end{bmatrix} \quad \checkmark$$

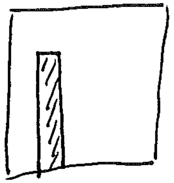
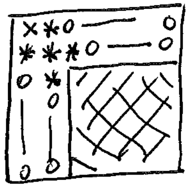
From A (dense) to T (tridiagonal)

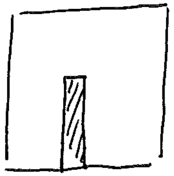
IDEA: $A =$  $x \in \mathbb{R}^{m-1}$ → $\underbrace{H(\cdot)}_{m-1\times m-1}x = \begin{bmatrix}*\\0\\\vdots\\0\end{bmatrix}$ → $\underbrace{Q}_{m\times m}\cdot A =$ 

→ $QAQ^T =$  → RECURSIVELY

UNBLOCKED ALGORITHM

$A^{(0)} = A \rightarrow v^{(0)} =$  $\rightarrow H_0, Q_0 \rightarrow A^{(1)} = Q_0 A^{(0)} Q_0^T =$ 

$A^{(1)} \rightarrow v^{(1)} =$  $\rightarrow H_1, Q_1 \rightarrow A^{(2)} = Q_1 A^{(1)} Q_1^T =$ 

$A^{(2)} \rightarrow v^{(2)} =$  $\rightarrow \ldots$

$$\underbrace{Q_{n-3} \cdots Q_1 Q_0}_{Q} A Q_0^T Q_1^T \cdots Q_{n-3}^T = T$$

$$Q \neq Q^T$$

$$Q^T Q = Q Q^T = I \rightarrow \text{UNITARY}$$

Blocked Algorithm (1)

· IDEA: $$Q_k \cdots Q_2 A Q_1^T \cdots Q_k^T = \underbrace{A + U_k W_k^T + W_k U_k^T}_{\text{SYR2K}} \equiv A^{(k)}$$

SYR2K, BLAS 3

$$U_k = \left[\, v_1 \,\middle|\, v_2 \,\middle|\, \cdots \,\middle|\, v_k \,\right] \in \mathbb{R}^{n\times k}$$

$$= \boxed{A} + \underbrace{\boxed{U_k}}_{k}\,\boxed{W_k^T} + \underbrace{\boxed{W_k}}_{k}\,\boxed{U_k^T}$$

$$W_k = \left[\quad ? \quad\right] \in \mathbb{R}^{n\times k}$$

(1) $$Q_{k+1} Q_k \cdots Q_1 A Q_1^T \cdots Q_k^T Q_{k+1}^T = A + U_{k+1} W_{k+1}^T + W_{k+1} U_{k+1}^T =$$

$$= A + \left[\, U_k \,\middle|\, v_{k+1} \,\right] \begin{bmatrix} W_k^T \\ \hline w_{k+1}^T \end{bmatrix} + \left[\, W_k \,\middle|\, w_{k+1} \,\right] \begin{bmatrix} U_k^T \\ \hline v_{k+1}^T \end{bmatrix} =$$

$$= A + U_k W_k^T + W_k U_k^T + v_{k+1} w_{k+1}^T + w_{k+1} v_{k+1}^T = A^{(k)} + v_{k+1} w_{k+1}^T + w_{k+1} v_{k+1}^T$$

(2) $$Q_{k+1} Q_k \cdots Q_1 A Q_1^T \cdots Q_k^T Q_{k+1}^T = Q_{k+1} A^{(k)} Q_{k+1}^T = \left(I - \beta v_{k+1} v_{k+1}^T\right) A^{(k)} \left(I - \beta v_{k+1} v_{k+1}^T\right) =$$

$$= A^{(k)} - \beta v_{k+1} v_{k+1}^T A^{(k)} - \beta A^{(k)} v_{k+1} v_{k+1}^T + \beta^2 v \underbrace{v^T A v}_{\in \mathbb{R}} v^T =$$

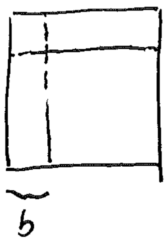
$$= A^{(k)} - v_{k+1} \cdot \beta v_{k+1}^T A^{(k)} + \frac{\beta^2}{2} v_{k+1}^T A^{(k)} v_{k+1} \cdot v_{k+1} v_{k+1}^T - \beta A^{(k)} v_{k+1} v_{k+1}^T + \frac{\beta^2}{2} v_{k+1}^T A^{(k)} v_{k+1} \cdot v_{k+1} v_{k+1}^T$$

$$= A^{(k)} - v_{k+1} \cdot \underbrace{\beta v_{k+1}^T \left(A^{(k)} - \frac{\beta}{2} v_{k+1}^T A^{(k)} v_{k+1} \cdot I\right)}_{w_{k+1}^T} - \underbrace{\beta \left(A^{(k)} - \frac{\beta}{2} v_{k+1}^T A^{(k)} v_{k+1} \cdot I\right) v_{k+1}}_{w_k} \cdot v_{k+1}^T$$

$$w_{k+1} := \beta\left(A^{(k)} - \beta/2\, v_{k+1}^T A^{(k)} v_{k+1} \cdot I\right) v_{k+1}$$

$$\Rightarrow \quad z_{k+1} := \beta A^{(k)} v_{k+1}, \qquad w_{k+1} := z - \beta/2\, \underbrace{z^T v}_{\in \mathbb{R}} v$$

BLOCKED ALGORITHM (2)

$A =$  $= \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$

$\longrightarrow$ REDUCE $\begin{bmatrix} A_{TL} \\ \hline A_{BL} \end{bmatrix}$ TO TRIDIAGONAL FORM, AND CONSTRUCT U AND W

$\longrightarrow$ APPLY U AND W TO A_{BR} : $\quad A_{BR} = \underbrace{A_{BR} + UW^T + WU^T}_{\text{SYR2K}}$