Introduction to Scientific Computing Languages

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Fourier Transform

Let $f : \mathbf{R} \to \mathbf{C}$ be an integrable function.
Its *Fourier Transform* F is defined as the function

$$F(t) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i x t} dx.$$

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Discrete Fourier Transform

Let $\{x_k\}_0^{n-1}$ be a sequence of complex numbers.
Its *Discrete Fourier Transform* (DFT) is defined as the sequence

$$X_k = \sum_{j=0}^{n-1} x_j e^{-i 2\pi \frac{k}{n} j}.$$

Fast Fourier Transform

One of the top-10 algorithms of the century

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- $F_m = e^{\frac{-2\pi i}{m} kj}, \quad k, j \in [0, 1, \ldots, m-1]$
- $F_m = F_m^T$ (symmetry)
- Matlab: `fft`, `ifft`, `fftshift`, ...

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Recursive Definition

$$F_m = P_{m,\frac{m}{k}} \left(I_k \otimes F_{\frac{m}{k}}\right) D_{k,\frac{m}{k}} \left(F_k \otimes I_{\frac{m}{k}}\right)$$

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Identity matrix

$$I_k = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}_{[k \times k]}$$

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Kronecker Product

$$A = \begin{bmatrix} a_{00} & a_{01} & \dots \\ a_{10} & a_{11} & \\ \vdots & & \ddots \\ & & \end{bmatrix}_{[m \times n]} , \quad A \otimes B = \begin{bmatrix} a_{00}B & a_{01}B & \dots \\ a_{10}B & a_{11}B & \\ \vdots & & \ddots \\ & & \end{bmatrix}_{[? \times ?]}$$

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Twiddle Factors

$$D_{k,\frac{m}{k}} = \begin{bmatrix} (D^0 = I) & 0 & & \cdots & 0 \\ 0 & D & & & \\ & & D^2 & & \\ \vdots & & & \ddots & \\ 0 & & & & D^{k-1} \end{bmatrix}_{[m \times m]}$$

$$D = \texttt{diag}(e^{\frac{-2\pi i}{m} j}), \quad j \in [0, \ldots, \frac{m}{k} - 1]$$

Definitions

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Permutation

$$P_{m,\frac{m}{k}} = \text{stride}$$

$$P_{m,\frac{m}{2}} \left[\begin{array}{cccccccc} \times & \times & \times & \times & \dots & \times & \times \\ \circ & \circ & \circ & \circ & \dots & \circ & \circ \\ + & + & + & + & \dots & & \\ \vdots & \vdots & & & & & \\ \hline \hat{\times} & \hat{\times} & \hat{\times} & \dots & & & \\ \hat{\circ} & \hat{\circ} & \hat{\circ} & \dots & & & \\ \hat{+} & \hat{+} & \hat{+} & \dots & & & \\ \vdots & \vdots & & & & & \end{array}\right]_{[m\times m]} \rightarrow \left[\begin{array}{ccccccc} \times & \times & \times & \times & \dots & \times & \times \\ \hat{\times} & \hat{\times} & \hat{\times} & \hat{\times} & \dots & \hat{\times} & \hat{\times} \\ \hline \circ & \circ & \circ & \circ & \dots & \circ & \circ \\ \hat{\circ} & \hat{\circ} & \hat{\circ} & \hat{\circ} & \dots & \hat{\circ} & \hat{\circ} \\ \hline + & + & + & + & \dots & & \\ \hat{+} & \hat{+} & \hat{+} & \hat{+} & \dots & & \\ \hline \vdots & \vdots & & & & & \end{array}\right]_{[m\times m]}$$

Definitions

$$F_m = P_{m,\frac{m}{k}} \left(I_k \otimes F_{\frac{m}{k}}\right) D_{k,\frac{m}{k}} \left(F_k \otimes I_{\frac{m}{k}}\right)$$

Permutation

$$P_{12,3} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \hline \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \hline \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \hline \alpha_9 \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} \rightarrow \begin{bmatrix} \alpha_0 \\ \alpha_3 \\ \alpha_6 \\ \alpha_9 \\ \hline \alpha_1 \\ \alpha_4 \\ \alpha_7 \\ \alpha_{10} \\ \hline \alpha_2 \\ \alpha_5 \\ \alpha_8 \\ \alpha_{11} \end{bmatrix} \qquad P_{12,2} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \hline \alpha_2 \\ \alpha_3 \\ \hline \alpha_4 \\ \alpha_5 \\ \hline \alpha_6 \\ \alpha_7 \\ \hline \alpha_8 \\ \alpha_9 \\ \hline \alpha_{10} \\ \alpha_{11} \end{bmatrix} \rightarrow \begin{bmatrix} \alpha_0 \\ \alpha_2 \\ \alpha_4 \\ \alpha_6 \\ \alpha_8 \\ \alpha_{10} \\ \hline \alpha_1 \\ \alpha_3 \\ \alpha_5 \\ \alpha_7 \\ \alpha_9 \\ \alpha_{11} \end{bmatrix}$$

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- 2-dimensional approach.

$$Y =$$

$$P_{m,\frac{m}{k}}(I_k \otimes F_{\frac{m}{k}})D_{k,\frac{m}{k}}(F_k \otimes I_{\frac{m}{k}})X(F_h \otimes I_{\frac{n}{h}})D_{h,\frac{n}{h}}(I_h \otimes F_{\frac{n}{h}})P^T_{n,\frac{n}{h}}$$

- Matlab: `fft2`, `ifft2`, ...