Knowledge-Based Automatic Generation of Partitioned Matrix Expressions

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Objective: generation of algorithms

$LL^T = A \quad \longrightarrow$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & \star \\ \hline A_{BL} & A_{BR} \end{array}\right)$
where A_{TL} is 0×0

While $n(A_{TL}) < n(A)$ **do**

$$\left(\begin{array}{c|c} A_{TL} & \star \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & \star & \star \\ \hline A_{10} & A_{11} & \star \\ \hline A_{20} & A_{21} & A_{22} \end{array}\right)$$

$$\begin{aligned} A_{11} &= \Gamma(A_{11}) \\ A_{21} &= A_{21} TRIL(A_{11})^{-T} \\ A_{22} &= A_{22} - A_{21} A_{21}^T \end{aligned}$$

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Target: Matrix operations

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- Correct by construction

AI ces

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- Correct by construction
- Loop invariants needed beforehand

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endwhile

- We aim at loop-based algorithms
- Correct by construction
- Loop invariants needed beforehand
- Loop invariants come from the PME

Example of PME

Cholesky Factorization

$$LL^T = A$$

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or in explicit form:

$$L := \Gamma(A)$$

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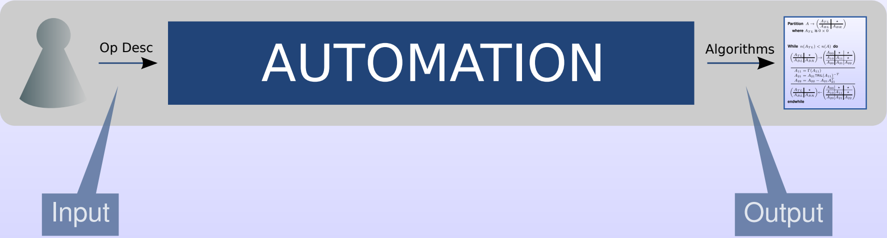
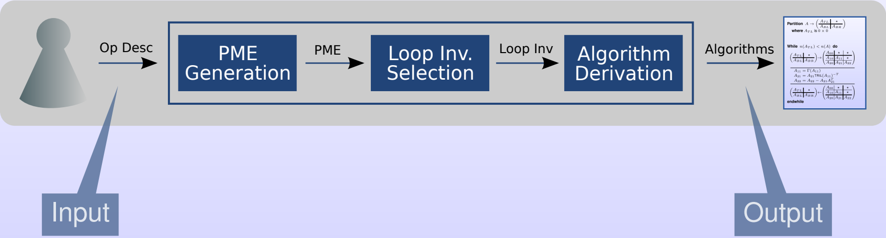
Partitioned Matrix Expression:

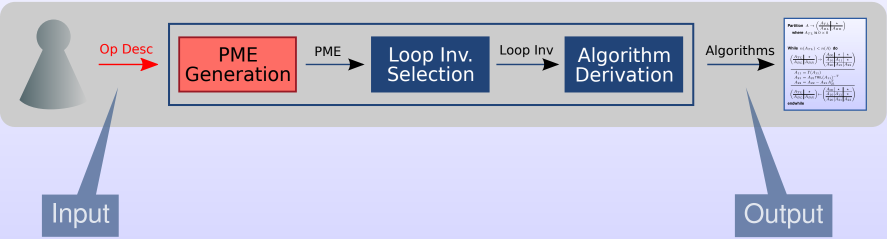
$$\left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} = A_{BL} L_{TL}^{-T} & L_{BR} = \Gamma(A_{BR} - L_{BL} L_{BL}^T) \end{array}\right)$$


Op Desc
AUTOMATION
Algorithms
Input
Output

CL1CK


Op Desc
PME Generation
PME
Loop Inv. Selection
Loop Inv
Algorithm Derivation
Algorithms
Input
Output

CL1CK


Op Desc
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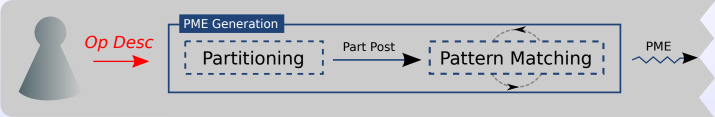
Output

Outline

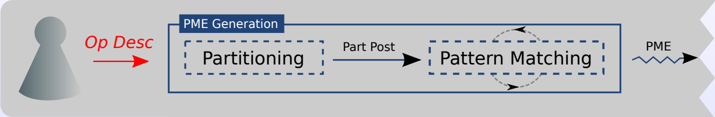
Operation Description

An example: Cholesky Factorization



$$LL^T = A \text{ ?}$$

An example: Cholesky Factorization



$$XY = Z \text{ ?}$$

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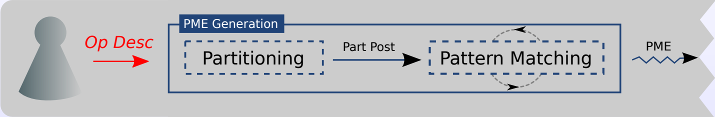
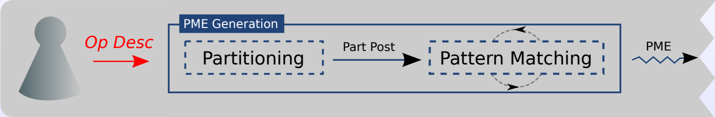


$$XY = Z \text{ ?}$$

$$Z \leftarrow XY$$



$$XY = Z\ ?$$

$$Z \leftarrow XY \qquad X \leftarrow ZY^{-1} \qquad Y \leftarrow X^{-1}Z$$

Operation Description

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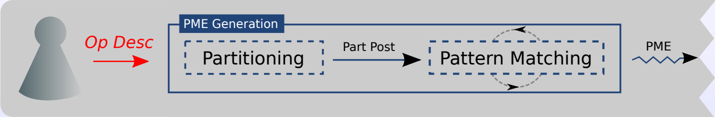

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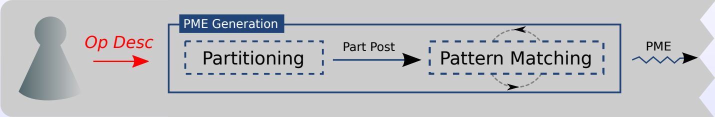

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- *Input* or *output* operand?

Operation Description

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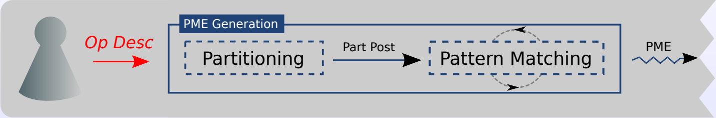
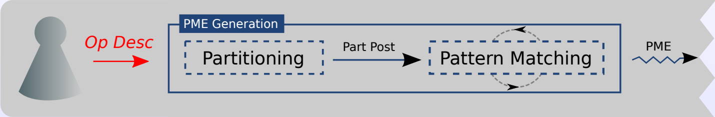


$$XY = Z \text{ ?}$$

$$Z \leftarrow XY \qquad X \leftarrow ZY^{-1} \qquad Y \leftarrow X^{-1}Z \qquad XY \leftarrow Z$$

- *Input* or *output* operand?
- Other properties: *lower triangular* ?, *symmetric* ?, . . .

An example: Cholesky Factorization

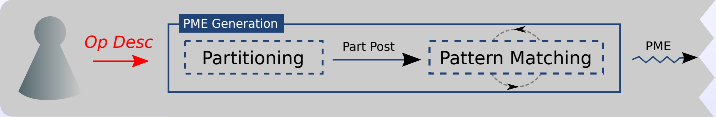


$$LL^T = A$$

$$\Downarrow$$

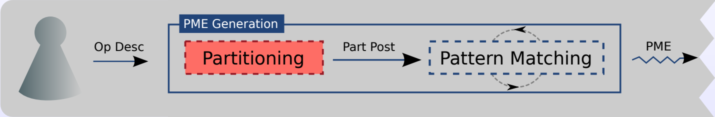
$$f: \quad L := \Gamma(A) \quad \equiv \begin{cases} f_{\text{Pre}}: & \{Input(A) \wedge SPD(A) \wedge \\ & \; Output(L) \wedge LowTri(L)\} \\ f_{\text{Post}}: & \{LL^T = A\} \end{cases}$$

An example: Cholesky Factorization



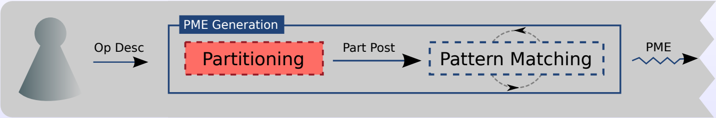

```
precond = {
    {L, {"Output", "Matrix", "LowerTriangular"}},
    {A, {"Input", "Matrix", "SPDLower"}}
  };
postcond = equal[times[L , trans[L]], A];
```

Outline

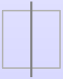

Op Desc
PME Generation
Partitioning
Part Post
Pattern Matching
PME

Partitioning

An example: Cholesky Factorization



$X \rightarrow$ 

$X \rightarrow$ 

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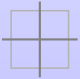
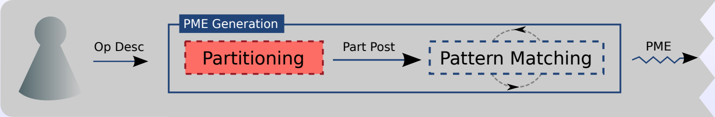
$X \rightarrow$ 

Partitioning

An example: Cholesky Factorization



$$X \rightarrow \left(X \right)$$

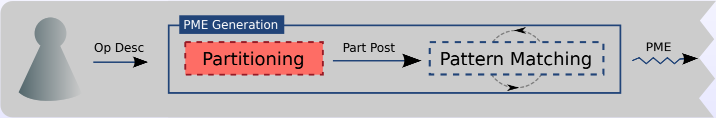
$$X \rightarrow \left(X_L \,\middle|\, X_R \right)$$

$$X \rightarrow \left(\begin{array}{c} X_T \\ \hline X_B \end{array} \right)$$

$$X \rightarrow \left(\begin{array}{c|c} X_{TL} & X_{TR} \\ \hline X_{BL} & X_{BR} \end{array} \right)$$

Partitioning

An example: Cholesky Factorization



$L \rightarrow$ 

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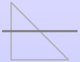
$L \rightarrow$ 

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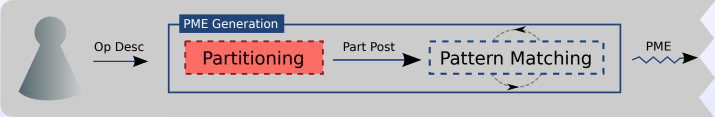
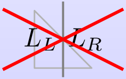
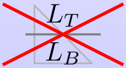


$L \rightarrow$ 

where L is lower triangular

$L \rightarrow$ 

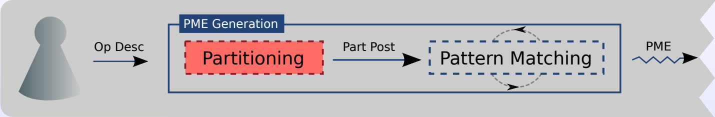
$L \rightarrow$ 

$$L \rightarrow \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)$$

where L_{TL} & L_{BR} are lower triangular

Cl1ck keeps track of the properties

An example: Cholesky Factorization



$$LL^T = A$$

L is **lower triangular:**  or $\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)$

A is **symmetric:**  or $\left(\begin{array}{c|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array}\right)$

Partitioning

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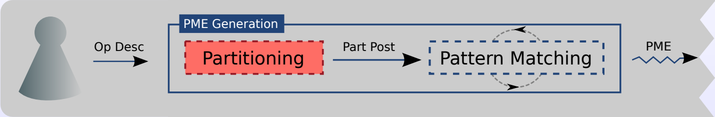

#	L	A	Partitioned Postcondition
1	$L \to (L)$	$A \to (A)$	$(L)\,(L)^T = (A)$
2	$L \to (L)$	$A \to \left(\begin{array}{c\|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array}\right)$	$(L)\,(L)^T = \left(\begin{array}{c\|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array}\right)$
3	$L \to \left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)$	$A \to (A)$	$\left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \left(\begin{array}{c\|c} L_{TL}^T & L_{BL}^T \\ \hline 0 & L_{BR}^T \end{array}\right) = (A)$
4	$L \to \left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)$	$A \to \left(\begin{array}{c\|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array}\right)$	$\left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \left(\begin{array}{c\|c} L_{TL}^T & L_{BL}^T \\ \hline 0 & L_{BR}^T \end{array}\right) = \left(\begin{array}{c\|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array}\right)$

Partitioning

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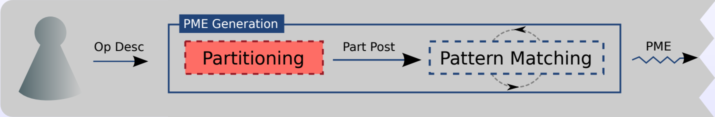

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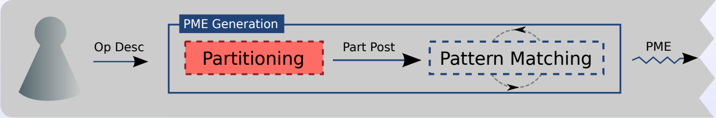
Non-Conformal Partitioning

An example: Cholesky Factorization



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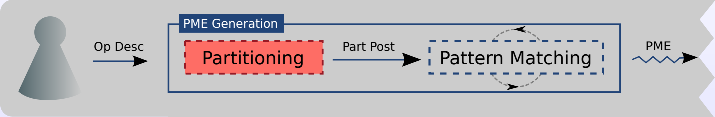
Does not decompose the operation



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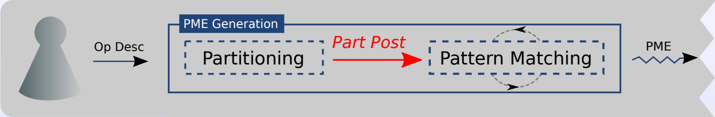
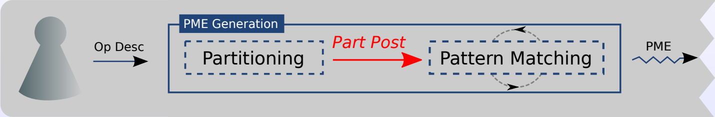
1	$L \rightarrow (L)$	$A \rightarrow (A)$	~~$(L)(L)^T = (A)$~~
2	$L \rightarrow (L)$	$A \rightarrow \left(\begin{array}{c\|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array}\right)$	~~$(L)(L)^T = \left(\begin{array}{c\|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array}\right)$~~
3	$L \rightarrow \left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)$	$A \rightarrow (A)$	~~$\left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)\left(\begin{array}{c\|c} L_{TL}^T & L_{BL}^T \\ \hline 0 & L_{BR}^T \end{array}\right) = (A)$~~
4	$L \rightarrow \left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)$	$A \rightarrow \left(\begin{array}{c\|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array}\right)$	$\left(\begin{array}{c\|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)\left(\begin{array}{c\|c} L_{TL}^T & L_{BL}^T \\ \hline 0 & L_{BR}^T \end{array}\right) = \left(\begin{array}{c\|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array}\right)$

Only the feasible ones are generated!



Partitioned postcondition:

$$\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)\left(\begin{array}{c|c} L_{TL}^T & L_{BL}^T \\ \hline 0 & L_{BR}^T \end{array}\right) = \left(\begin{array}{c|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array}\right)$$

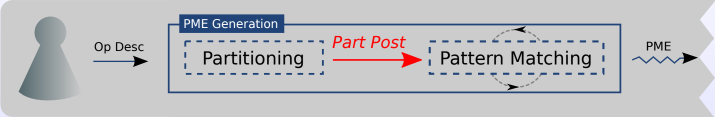


Symbolic computation:

$$\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)\left(\begin{array}{c|c} L_{TL}^T & L_{BL}^T \\ \hline 0 & L_{BR}^T \end{array}\right) = \left(\begin{array}{c|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array}\right)$$

$$\Downarrow$$

$$\left(\begin{array}{c|c} L_{TL}L_{TL}^T = A_{TL} & \star \\ \hline L_{BL}L_{TL}^T = A_{BL} & L_{BL}L_{BL}^T + L_{BR}L_{BR}^T = A_{BR} \end{array}\right)$$



Symbolic computation:

$$\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)\left(\begin{array}{c|c} L_{TL}^T & L_{BL}^T \\ \hline 0 & L_{BR}^T \end{array}\right) = \left(\begin{array}{c|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array}\right)$$

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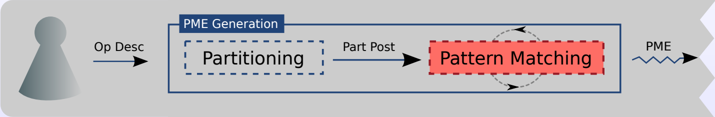
Not yet a PME!

An example: Cholesky Factorization



Canonical Form (Input / Output):

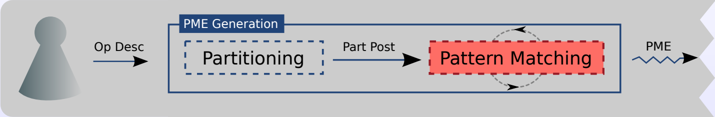
(1)
$$\left(\begin{array}{c|c} L_{TL}L_{TL}^T = A_{TL} & \star \\ \hline L_{BL}L_{TL}^T = A_{BL} & L_{BL}L_{BL}^T + L_{BR}L_{BR}^T = A_{BR} \end{array}\right)$$

$$\Downarrow$$

(2)
$$\left(\begin{array}{c|c} L_{TL}L_{TL}^T = A_{TL} & \star \\ \hline L_{BL}L_{TL}^T = A_{BL} & L_{BL}L_{BL}^T + L_{BR}L_{BR}^T = A_{BR} \end{array}\right)$$

Pattern Matching

An example: Cholesky Factorization

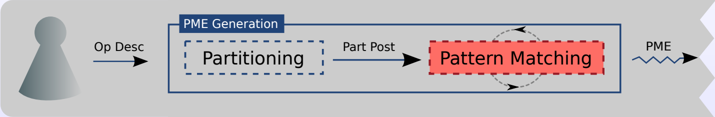


Pattern matching and algebraic manipulation:

$$(2) \quad \left(\begin{array}{c|c} L_{TL} L_{TL}^T = A_{TL} & \star \\ \hline L_{BL} L_{TL}^T = A_{BL} & L_{BL} L_{BL}^T + L_{BR} L_{BR}^T = A_{BR} \end{array} \right)$$

$$\Downarrow$$

$$(3) \quad \left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} L_{TL}^T = A_{BL} & L_{BL} L_{BL}^T + L_{BR} L_{BR}^T = A_{BR} \end{array} \right)$$

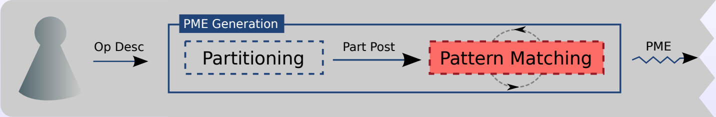


Pattern matching and algebraic manipulation:

$$(3) \quad \left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} L_{TL}^T = A_{BL} & L_{BL} L_{BL}^T + L_{BR} L_{BR}^T = A_{BR} \end{array} \right)$$

$$\Downarrow$$

$$(4) \quad \left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} L_{TL}^T = A_{BL} & L_{BL} L_{BL}^T + L_{BR} L_{BR}^T = A_{BR} \end{array} \right)$$



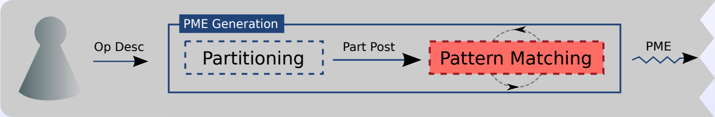
Pattern matching and algebraic manipulation:

$$(4) \quad \left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} L_{TL}^{T} = A_{BL} & L_{BL} L_{BL}^{T} + L_{BR} L_{BR}^{T} = A_{BR} \end{array} \right)$$

$$\Downarrow$$

$$(5) \quad \left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} = \mathtt{TRSM}(A_{BL}, L_{TL}^{-T}) & L_{BL} L_{BL}^{T} + L_{BR} L_{BR}^{T} = A_{BR} \end{array} \right)$$

An example: Cholesky Factorization

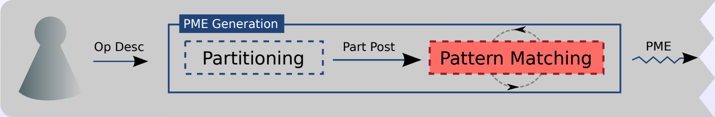


Pattern matching and algebraic manipulation:

$$(5)\quad \left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} = \texttt{TRSM}(A_{BL}, L_{TL}^{-T}) & L_{BL}L_{BL}^{T} + L_{BR}L_{BR}^{T} = A_{BR} \end{array}\right)$$

$$\Downarrow$$

$$(6)\quad \left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} = \texttt{TRSM}(A_{BL}, L_{TL}^{-T}) & L_{BL}L_{BL}^{T} + L_{BR}L_{BR}^{T} = A_{BR} \end{array}\right)$$



Pattern matching and algebraic manipulation:

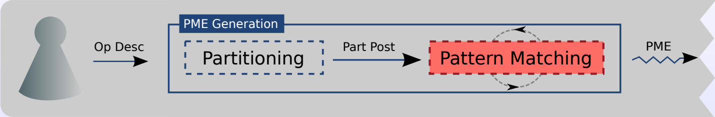
$$(6)\quad \left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} = \texttt{TRSM}(A_{BL}, L_{TL}^{-T}) & L_{BL}L_{BL}^T + L_{BR}L_{BR}^T = A_{BR} \end{array}\right)$$

$$\Downarrow$$

$$(7)\quad \left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} = \texttt{TRSM}(A_{BL}, L_{TL}^{-T}) & L_{BR}L_{BR}^T = A_{BR} - L_{BL}L_{BL}^T \end{array}\right)$$

Pattern Matching

An example: Cholesky Factorization



Pattern matching and algebraic manipulation:

$$\left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} = \texttt{TRSM}(A_{BL}, L_{TL}^{-T}) & {\color{red} L_{BR} L_{BR}^{T}} = A_{BR} - L_{BL} L_{BL}^{T} \end{array} \right) \tag{7}$$

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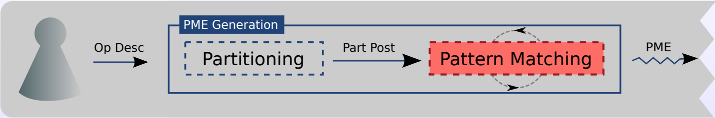
An example: Cholesky Factorization



Pattern matching and algebraic manipulation:

$$(7) \quad \left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} = \texttt{TRSM}(A_{BL}, L_{TL}^{-T}) & L_{BR}L_{BR}^{T} = A_{BR} - L_{BL}L_{BL}^{T} \end{array} \right)$$

$$\boxed{SPD(A_{BR} - L_{BL}L_{BL}^{T}) \ ?}$$

$$SPD(A_{BR} - L_{BL}L_{BL}^T) \; ?$$

Cholesky Theorem

$$A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array} \right)$$

$$SPD(A) \Longrightarrow \begin{cases} SPD(A_{TL}) \wedge \\ SPD(A_{BR} - A_{BL}A_{TL}^{-1}A_{BL}^T) \end{cases}$$

$$SPD(A_{BR} - L_{BL}L_{BL}^T)\ ?$$

Cholesky Theorem

$$A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array}\right)$$

$$SPD(A) \Longrightarrow \begin{cases} SPD(A_{TL}) \wedge \\ SPD(A_{BR} - A_{BL}A_{TL}^{-1}A_{BL}^T) \end{cases}$$

$$A_{BR} - L_{BL}L_{BL}^T \equiv A_{BR} - A_{BL}A_{TL}^{-1}A_{BL}^T\ ?$$

$$SPD(A_{BR} - L_{BL}L_{BL}^T)\ ?$$

Cholesky Theorem

$$A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array}\right)$$

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$$A_{BR} - L_{BL}L_{BL}^T \equiv A_{BR} - A_{BL}A_{TL}^{-1}A_{BL}^T\ ?$$

$$L_{BL} \rightarrow A_{BL}L_{TL}^{-T} \ ; \ L_{TL}L_{TL}^T \rightarrow A_{TL}$$

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$$SPD(A_{BR} - L_{BL}L_{BL}^T)\ ?$$

Cholesky Theorem

$$A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array}\right)$$

$$SPD(A) \Longrightarrow \begin{cases} SPD(A_{TL}) \wedge \\ SPD(A_{BR} - A_{BL}A_{TL}^{-1}A_{BL}^T) \end{cases}$$

$$A_{BR} - L_{BL}L_{BL}^T \equiv A_{BR} - A_{BL}A_{TL}^{-1}A_{BL}^T\ ?$$

$$L_{BL} \rightarrow A_{BL}L_{TL}^{-T} \quad ; \quad L_{TL}L_{TL}^T \rightarrow A_{TL}$$

$$L_{BL}L_{BL}^T \equiv A_{BL}L_{TL}^{-T}L_{TL}^{-1}A_{BL}^T$$

AI ces

SPD

$$SPD(A_{BR} - L_{BL} L_{BL}^T) \ ?$$

Cholesky Theorem

$$A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array} \right)$$

$$SPD(A) \Longrightarrow \begin{cases} SPD(A_{TL}) \wedge \\ SPD(A_{BR} - A_{BL} A_{TL}^{-1} A_{BL}^T) \end{cases}$$

$$A_{BR} - L_{BL} L_{BL}^T \equiv A_{BR} - A_{BL} A_{TL}^{-1} A_{BL}^T \ ?$$

$$L_{BL} \rightarrow A_{BL} L_{TL}^{-T} \quad ; \quad L_{TL} L_{TL}^T \rightarrow A_{TL}$$

$$L_{BL} L_{BL}^T \equiv A_{BL} L_{TL}^{-T} L_{TL}^{-1} A_{BL}^T \equiv A_{BL} (L_{TL} L_{TL}^T)^{-1} A_{BL}^T$$

$$SPD(A_{BR} - L_{BL}L_{BL}^T)\ ?$$

Cholesky Theorem

$$A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{BL}^T \\ \hline A_{BL} & A_{BR} \end{array}\right)$$

$$SPD(A) \Longrightarrow \begin{cases} SPD(A_{TL}) \wedge \\ SPD(A_{BR} - A_{BL}A_{TL}^{-1}A_{BL}^T) \end{cases}$$

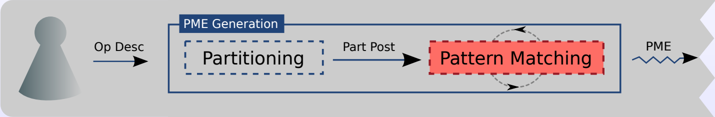
$$A_{BR} - L_{BL}L_{BL}^T \equiv A_{BR} - A_{BL}A_{TL}^{-1}A_{BL}^T\ ?$$

$$L_{BL} \rightarrow A_{BL}L_{TL}^{-T}\ ;\ L_{TL}L_{TL}^T \rightarrow A_{TL}$$

$$L_{BL}L_{BL}^T \equiv A_{BL}L_{TL}^{-T}L_{TL}^{-1}A_{BL}^T \equiv A_{BL}(L_{TL}L_{TL}^T)^{-1}A_{BL}^T \equiv A_{BL}A_{TL}^{-1}A_{BL}^T$$

Pattern Matching

An example: Cholesky Factorization

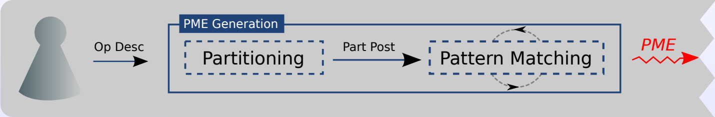


Pattern matching and algebraic manipulation:

$$(7) \quad \left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} = \texttt{TRSM}(A_{BL}, L_{TL}^{-T}) & L_{BR} L_{BR}^{T} = A_{BR} - L_{BL} L_{BL}^{T} \end{array} \right)$$

$$\Downarrow$$

$$(8) \quad \left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} = \texttt{TRSM}(A_{BL}, L_{TL}^{-T}) & L_{BR} = \Gamma(A_{BR} - L_{BL} L_{BL}^{T}) \end{array} \right)$$

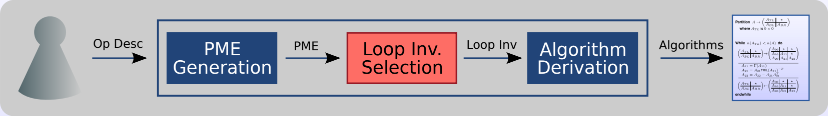


Partitioned Matrix Expression:

$$\left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} = A_{BL} L_{TL}^{-T} & L_{BR} = \Gamma(A_{BR} - L_{BL} L_{BL}^{T}) \end{array}\right)$$

Outline

An example: Cholesky Factorization



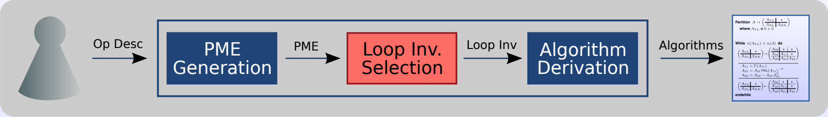
Partitioned Matrix Expression:

$$\left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} = A_{BL} L_{TL}^{-T} & L_{BR} = \Gamma(A_{BR} - L_{BL} L_{BL}^{T}) \end{array}\right)$$

- Decomposition of the problem.
- Computation to be performed.

Loop Invariants

An example: Cholesky Factorization



Partitioned Matrix Expression:

$$\left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} = A_{BL} L_{TL}^{-T} & L_{BR} = \Gamma(A_{BR} - L_{BL} L_{BL}^{T}) \end{array}\right)$$

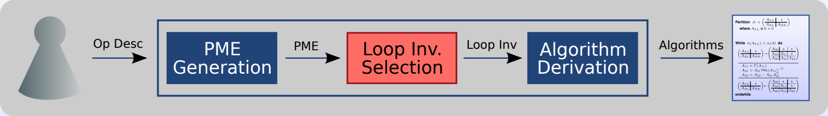
- Decomposition of the problem.
- Computation to be performed.

Loop invariants:

- Subset of the total computation.
- Not all subsets are valid.

Loop Invariants

An example: Cholesky Factorization



Loop Invariants:

- $$\left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} = 0 & L_{BR} = 0 \end{array}\right)$$
- $$\left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} = A_{BL} L_{TL}^{-T} & L_{BR} = 0 \end{array}\right)$$
- $$\left(\begin{array}{c|c} L_{TL} = \Gamma(A_{TL}) & \star \\ \hline L_{BL} = A_{BL} L_{TL}^{-T} & L_{BR} = A_{BR} - L_{BL} L_{BL}^{T} \end{array}\right)$$

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An example: Cholesky Factorization

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & \star \\ \hline A_{BL} & A_{BR} \end{array}\right)$
where A_{TL} is 0×0

while $n(A_{TL}) < n(A)$ **do**

$$\left(\begin{array}{c|c} A_{TL} & \star \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & \star & \star \\ \hline A_{10} & A_{11} & \star \\ \hline A_{20} & A_{21} & A_{22} \end{array}\right)$$

Variant 1

$$L_{10} = A_{10} L_{00}^{-T}$$
$$L_{11} = A_{11} - L_{10} L_{10}^{T}$$
$$L_{11} = \Gamma(L_{11})$$

Variant 2

$$L_{11} = A_{11} - L_{10} L_{10}^{T}$$
$$L_{11} = \Gamma(L_{11})$$
$$L_{21} = A_{21} - L_{20} L_{10}^{T}$$
$$L_{21} = L_{21} L_{11}^{-T}$$

Variant 3

$$L_{11} = \Gamma(L_{11})$$
$$L_{21} = L_{21} L_{11}^{-T}$$
$$L_{22} = L_{22} - L_{21} L_{21}^{T}$$

$$\left(\begin{array}{c|c} A_{TL} & \star \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & \star & \star \\ \hline A_{10} & A_{11} & \star \\ \hline A_{20} & A_{21} & A_{22} \end{array}\right)$$

endwhile

* Variant 3 needs the assignment $L_{BR} = A_{BR}$ before entering the loop.

AI ces

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & \star \\ \hline A_{BL} & A_{BR} \end{array}\right)$

where A_{TL} is 0×0

While $n(A_{TL}) < n(A)$ **do**

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$$\begin{aligned} A_{11} &= \Gamma(A_{11}) \\ A_{21} &= A_{21} TRIL(A_{11})^{-T} \\ A_{22} &= A_{22} - A_{21} A_{21}^T \end{aligned}$$

$$\left(\begin{array}{c|c} A_{TL} & \star \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & \star & \star \\ \hline A_{10} & A_{11} & \star \\ \hline A_{20} & A_{21} & A_{22} \end{array}\right)$$

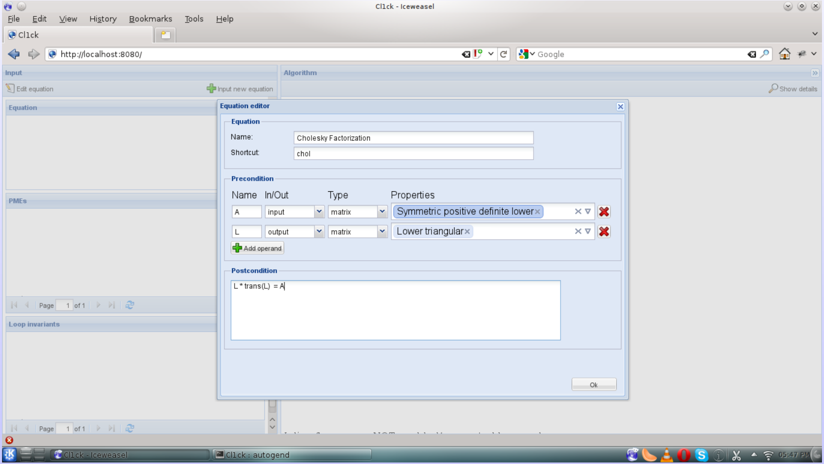
endwhile

$\Longleftrightarrow$

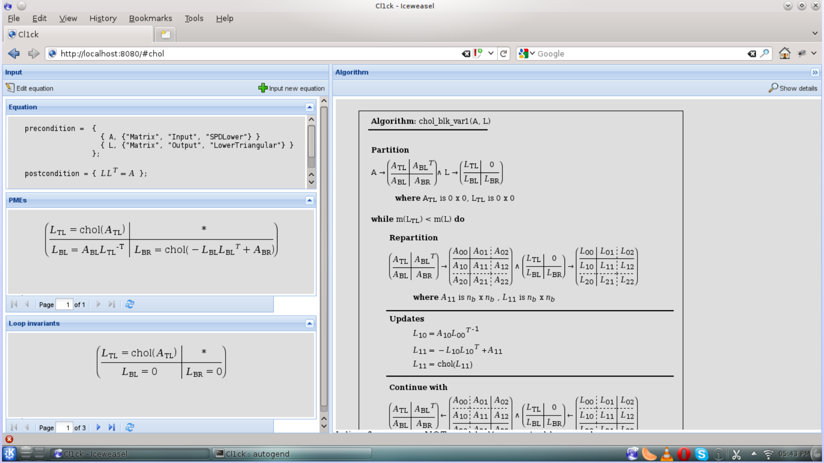
```
FLA_Part_2x2( A,      &ATL, &ATR,
                      &ABL, &ABR,     0, 0, FLA_TL );

while ( FLA_Obj_length( ATL ) < FLA_Obj_length( A ) )
{
   b = min( FLA_Obj_length( ABR ), nb_alg );
   FLA_Repart_2x2_to_3x3(
      ATL, /**/ ATR,       &A00, /**/ &A01, &A02,
   /* ************* */   /* ******************** */
                           &A10, /**/ &A11, &A12,
      ABL, /**/ ABR,       &A20, /**/ &A21, &A22,
      b, b, FLA_BR );
   /*-----------------------------------------------*/
    FLA_Chol( FLA_LOWER_TRIANGULAR, A11 );
    FLA_Trsm( FLA_RIGHT, FLA_LOWER_TRIANGULAR,
              FLA_TRANSPOSE, FLA_NONUNIT_DIAG,
              FLA_ONE, A11, A21 );
    FLA_Syrk( FLA_LOWER_TRIANGULAR, FLA_NO_TRANSPOSE,
              FLA_MINUS_ONE, A21, FLA_ONE, A22 );
   /*-----------------------------------------------*/
   FLA_Cont_with_3x3_to_2x2(
      &ATL, /**/ &ATR,       A00, A01, /**/ A02,
                             A10, A11, /**/ A12,
    /* ************** */   /* ****************** */
      &ABL, /**/ &ABR,       A20, A21, /**/ A22,
      FLA_TL );
}
```

An example: Cholesky Factorization



An example: Cholesky Factorization



Outline

Conclusions

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- Minimum amount of knowledge:
  - Input and output operands
  - Structure of the operands: triangularity, symmetry, . . .

Conclusions

- Minimum amount of knowledge:
  - Input and output operands
  - Structure of the operands: triangularity, symmetry, ...
- Knowledge implemented in Cl1ck:
  - Basic matrix algebra
  - Pattern matching
  - Properties inheritance
  - Theorems

Conclusions

- Minimum amount of knowledge:
  - Input and output operands
  - Structure of the operands: triangularity, symmetry, . . .
- Knowledge implemented in Cl1ck:
  - Basic matrix algebra
  - Pattern matching
  - Properties inheritance
  - Theorems
- First prototype for the fully automatic generation of algorithms

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- Dr. Edoardo Di Napoli
- Matthias Petschow
- Roman Iakymchuk

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