A Modular and Systematic Approach to Stability Analysis

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AICES

Idea: error analysis ≡ matrix equation

- $LU = A \quad \Rightarrow \quad \check{L}\check{U} = A + \Delta A$;
  ΔA unknown, "error variable".
  Error analysis ≡ computing ΔA.
- $|\Delta A| \leq O(n)\mathbf{u}|\check{L}||\check{U}|$,
  $|\Delta A| \leq O(n)\mathbf{u}(|\check{L}||\check{U}| + |A|)$?

• **Step #1:** From algorithm to loop invariant.

Algorithm: LU

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$
where A_{TL} is 0×0
While $size(A_{TL}) < size(A)$ **do**
Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

$\alpha_{11} := \alpha_{11} - a_{10}^T a_{01}$ DOT(1)
$a_{12}^T := (a_{12}^T - a_{10}^T A_{02})$ GEMV(1)
$a_{21} := (a_{21} - A_{20} a_{01})/\alpha_{11}$ GEMV(2)

Continue

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

endwhile

Algorithm Progression

DONE	DONE	DONE
DONE	DOT	GEMV
DONE	GEMV SCAL	

Loop Invariant

$$\left(\begin{array}{c|c} L_{TL}U_{TL} = A_{TL} & L_{TL}U_{TR} = A_{TR} \\ \hline L_{BL}U_{TL} = A_{BL} & — \end{array}\right)$$

• **Step #2:** From loop invariant to error invariant.

Error Invariant: $\left(\begin{array}{c|c} \check{L}_{TL}\check{U}_{TL} = A_{TL} + \Delta A_{TL} & \check{L}_{TL}\check{U}_{TR} = A_{TR} + \Delta A_{TR} \\ \hline \check{L}_{BL}\check{U}_{TL} = A_{BL} + \Delta A_{BL} & — \end{array}\right)$

• **Step #3:** From error invariant to subgoals.

The error invariant is expressed in terms of the repartitioned operands.

$$\left\{\left(\begin{array}{c|c|c} \check{L}_{00}\check{U}_{00} = A_{00} + \Delta A_{00} & \check{L}_{00}\check{u}_{01} = a_{01} + \delta a_{01} & \check{L}_{00}\check{U}_{02} = A_{02} + \Delta A_{02} \\ \hline \check{l}_{10}^T\check{U}_{00} = a_{10}^T + \delta a_{10}^T & — & — \\ \hline \check{L}_{20}\check{U}_{00} = A_{20} + \Delta A_{02} & — & — \end{array}\right)\right\}$$

$v_{11} := \alpha_{11} - l_{10}^T u_{01}$ DOT(1)
$u_{12}^T := a_{12}^T - l_{10}^T U_{02}$ GEMV(1)
$l_{21} := (a_{21} - L_{20} u_{01})/v_{11}$ GEMV(2)

Error Updates

$$\left\{\left(\begin{array}{c|c|c} \check{L}_{00}\check{U}_{00} = A_{00} + \Delta A_{00} & \check{L}_{00}\check{u}_{01} = a_{01} + \delta a_{01} & \check{L}_{00}\check{U}_{02} = A_{02} + \Delta A_{02} \\ \hline \check{l}_{10}^T\check{U}_{00} = a_{10}^T + \delta a_{10}^T & \check{l}_{10}^T\check{u}_{01} + \check{v}_{11} = \alpha_{11} + \delta\alpha_{11} & \check{l}_{10}^T\check{U}_{02} + \check{u}_{12}^T = a_{12}^T + \delta a_{12}^T \\ \hline \check{L}_{20}\check{U}_{00} = A_{20} + \Delta A_{20} & \check{L}_{20}\check{u}_{01} + \check{l}_{21}\check{v}_{11} = a_{21} + \delta a_{21} & — \end{array}\right)\right\}$$

• **Step #4:** Inventory of known results or new analysis.

Can the error generated at each iteration be accumulated into the error variables ($\delta\alpha_{11}$, δa_{01}, and δa_{12}^T) so that the error invariant remains satisfied?

Subgoals

- DOT: $v_{11} := \alpha_{11} - l_{10}^T u_{01} \quad \overset{?}{\Rightarrow} \quad \check{l}_{10}^T\check{u}_{01} + \check{v}_{11} = \alpha_{11} + \delta\alpha_{11}$
- GEMV: $u_{12}^T := a_{12}^T - l_{10}^T U_{02} \quad \overset{?}{\Rightarrow} \quad \check{l}_{10}^T\check{U}_{02} + \check{u}_{12}^T = a_{12}^T + \delta a_{12}^T$
- GEMV: $l_{21} := (a_{21} - L_{20} u_{01})/v_{11} \quad \overset{?}{\Rightarrow} \quad \check{L}_{20}\check{u}_{01} + \check{l}_{21}\check{v}_{11} = a_{21} + \delta a_{21}$

DOT: $\nu := (\beta - l^T x)/\lambda$ Inventory

Results built directly from the computational models

1: $\lambda\check{\nu} = (\beta + \delta\beta) - (l + \delta l)^T x \quad \begin{cases} |\delta\beta| \leq \gamma_2|\beta| \\ |\delta l| \leq \gamma_{n+2}|l| \end{cases}$

2: $(\lambda + \delta\lambda)\check{\nu} = \beta - (l + \delta l)^T x \quad \begin{cases} |\delta\lambda| \leq \gamma_2|\lambda| \\ |\delta l| \leq \gamma_n|l| \end{cases}$

3: $(\lambda + \delta\lambda)\check{\nu} = (\beta + \delta\beta) - (l + \delta l)^T x \quad \begin{cases} |\delta\beta| \leq \gamma_1|\beta| \\ |\delta l| \leq \gamma_{n+1}|l| \\ |\delta\lambda| \leq \gamma_1|\lambda| \end{cases}$

GEMV: $w := (y - Ax)/\lambda$

Results built from those of DOT and Axpy.

• **Step #5:** Inductive step.

A Goal-Oriented and Modular Approach to Stability Analysis.
P. Bientinesi and R. van de Geijn. SIMAX, 32(1), March 2011.

Application: fine-grain bounds

$Lx = b \Rightarrow \begin{cases} (L + \Delta L)\check{x} = b \\ |\Delta L| \leq \gamma_n|L| \end{cases}$, i.e., $|\Delta L| \leq \begin{bmatrix} \gamma_n & & 0 \\ \vdots & \ddots & \\ \gamma_n & \cdots & \gamma_n \end{bmatrix} \odot |L|$

$\odot$ = pointwise multiplication

For some applications the factor γ_n is too coarse.
The interest lays on how different variables affect/are affected by the computation.

Example: **Algorithm** TRSV

Partition $L \rightarrow \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right), x \rightarrow \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right), b \rightarrow \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right)$
where L_{TL}, x_T, and b_T are empty
While $m(x_T) < m(x)$ **do**
Repartition

$$\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right), \ldots$$

$\chi_1 := (\beta_1 - l_{10}^T x_0)/\lambda_{11}$ DOT(2)

Continue

$$\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right), \ldots$$

endwhile

Theorem: $|\Delta L| \leq T \odot |L|$,
where $T \leq \text{LowTriToepliz}(\{\gamma_2, \gamma_2, \gamma_3, \ldots, \gamma_n\}^T)$.
Proof: Using a different analysis for DOT. $[x^T y] = x^T y + x^T \Sigma^{(n)} y$,
where $\Sigma^{(n)} \leq \text{diag}(\{\gamma_n, \gamma_n, \gamma_{n-1}, \ldots, \gamma_2\})$. Then

$$T = \begin{bmatrix} \gamma_2 & & & & & \\ \gamma_1 & \gamma_2 & & & & \\ \gamma_2 & \gamma_2 & \gamma_2 & & & \\ \gamma_3 & \gamma_3 & \gamma_2 & \gamma_2 & & \\ \vdots & & & & \ddots & \\ \gamma_{n-1} & \gamma_{n-1} & \gamma_{n-2} & \cdots & \gamma_2 & \gamma_2 \end{bmatrix}$$