The Landscape of High-Performance Tensor Contractions

Paul Springer and Paolo Bientinesi

Aachen Institute for Advanced Study in
Computational Engineering Science

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  - 0-order tensor: scalar α

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- Tensor contractions can be thought of as generalized GEMMs
- Three approaches to tensor contractions:
  - Nested loops
  - Loops over GEMM (LoG)
  - Transpose-Transpose-GEMM-Transpose (TTGT)

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- Three approaches to tensor contractions:
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  - Loops over GEMM (LoG)
  - Transpose-Transpose-GEMM-Transpose (TTGT)
- We propose a novel approach: GETT[1]
  - Akin to a high-performance GEMM implementation

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Outline

- Approaches to Tensor Contractions:
  - Loops over GEMM (LoG)
  - Transpose-Transpose-GEMM-Transpose (TTGT)
  - GEMM-like Tensor-Tensor Multiply (GETT)
- Tensor Contraction Code Generator[2]
- Performance Evaluation

[2] Source code available at: `https://github.com/HPAC/tccg`

Conceptual Idea

Identify 2D subtensors and contract them via GEMM

- $\mathcal{C}_{m_1,n_1} \leftarrow \sum_{k_1} \mathcal{A}_{m_1,k_1} \mathcal{B}_{k_1,n_1}$

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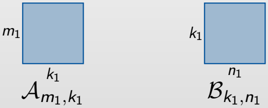


```
gemm(M₁, N₁, K₁, A[:,:], B[:,:], C[:,:])
```

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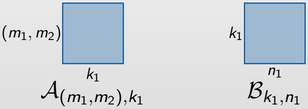

gemm($M_1 \times M_2$, N_1, K_1, $\mathcal{A}[:,:]$, $\mathcal{B}[:,:]$, $\mathcal{C}[:,:]$)

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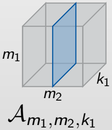

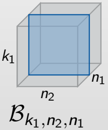


```
for( m2 = 0; m2 < M2; m2++ )
  for( n1 = 0; n1 < N1; n1++ )
    gemm(M1, N2, K1, A[:,m2,:], B[:,:,n1], C[:,n1,:,m2])
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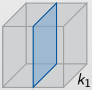

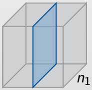


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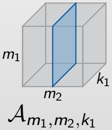

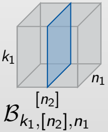


```
for( m₂ = 0; m₂ < M₂; m₂++ )
  gemm_batch(M₁, N₁, K₁, A[:, m₂, :], B[:, n₂, :], C[:, :, n₂, m₂], N₂)
```

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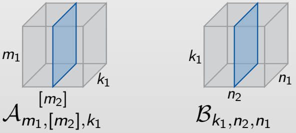


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for( n2 = 0; n2 < N2; n2++ )
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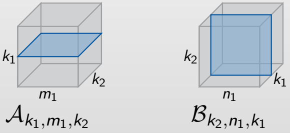


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for( k1 = 0; k1 < K1; k1++ )
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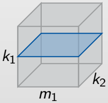
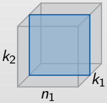


$\mathcal{A}_{k_1,m_1,k_2}$



$\mathcal{B}_{k_2,n_1,k_1}$

```
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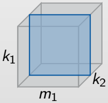

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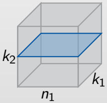

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```
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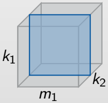

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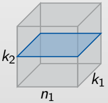

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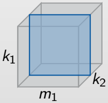

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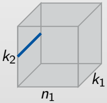

$\mathcal{B}_{k_2,n_1,k_1}$

```
for( n = 0; n1 < N1; n1++ )
  for( k2 = 0; k2 < K2; k2++ )
    gemv(M1, K1, A[:,:,k2]^T, B[k2,n1,:], C[:,n])
```

- Search space:

Loop Over GEMM (LoG)

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- Disadvantages:
  - Some contractions cannot be implemented via straight LoG
  - GEMM's arithmetic intensity can be suboptimal

- Free indices of $\mathcal{A}$
  - $I_m := \{m_1, m_2, ..., m_\gamma\} = I_{\mathcal{A}} \cap I_{\mathcal{C}}$

[3] Di Napoli et al. "Towards an Efficient Use of the BLAS Library for Multilinear Tensor Contractions"
[4] Yang Shi et al. "Tensor Contractions with Extended BLAS Kernels on CPU and GPU"

Map Tensor Contractions to BLAS

- Free indices of $\mathcal{A}$
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- Free indices of $\mathcal{B}$
  - $I_n := \{n_1, n_2, ..., n_\zeta\} = I_\mathcal{B} \cap I_\mathcal{C}$

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- Tensor contractions can be mapped to BLAS routines[3,4]:
  - `GEMM`: $I_m \neq \emptyset$ **and** $I_n \neq \emptyset$ **and** $I_k \neq \emptyset$.

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  - `GEMV`: $(I_m = \emptyset$ **or** $I_n = \emptyset)$ **and** $I_k \neq \emptyset$
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- Tensor contractions can be mapped to BLAS routines[3,4]:
  - `GEMM`: $I_m \neq \emptyset$ **and** $I_n \neq \emptyset$ **and** $I_k \neq \emptyset$.
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  - `GER`: $I_m \neq \emptyset$ **and** $I_n \neq \emptyset$ **and** $I_k = \emptyset$
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[3] Di Napoli et al. “Towards an Efficient Use of the BLAS Library for Multilinear Tensor Contractions”

[4] Yang Shi et al. “Tensor Contractions with Extended BLAS Kernels on CPU and GPU”

- Free indices of $\mathcal{A}$
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AICES

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... and more.

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[5] Paul Springer et al. "TTC: A high-performance Compiler for Tensor Transpositions"

Key Idea

- Eliminate explicit transpositions
- Pack-and-transpose while moving data into the caches[5]
  - $\Rightarrow$ Complexity offloaded into packing routines

[5] Sharan Chetlur et al. "CUDNN: Efficient Primitives for Deep Learning"

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```
// N-Loop
for $n$ = 1 : $nc$ : $S_{I_n}$
   // K-Loop (contracted)
   for $k$ = 1 : $kc$ : $S_{I_k}$
      $\widehat{\mathcal{B}}$ = identify_subtensor($B, n, k$)
      // pack $\widehat{\mathcal{B}}$ into $\widetilde{\mathcal{B}}$ (L3 cache)
      $\widetilde{\mathcal{B}}$ = packB($\widehat{\mathcal{B}}$)
      // M-Loop
      for $m$ = 1 : $mc$ : $S_{I_m}$
         $\widehat{\mathcal{A}}$ = identify_subtensor($A, m, k$)
         // pack $\widehat{\mathcal{A}}$ into $\widetilde{\mathcal{A}}$ (L2 cache)
         $\widetilde{\mathcal{A}}$ = packA($\widehat{\mathcal{A}}$)
         $\widehat{\mathcal{C}}$ = identify_subtensor($C, m, n$)
         // compute matrix-matrix product of $\widetilde{\mathcal{A}}\widetilde{\mathcal{B}}$
         macroKernel($\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widehat{\mathcal{C}}, \alpha, \beta$)
```

[5] Sharan Chetlur et al. "CUDNN: Efficient Primitives for Deep Learning"

GEMM-like Tensor-Tensor Multiplication (GETT)

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```
// N-Loop
for n = 1 : nc : S_{I_n}
```

Monday, February 27 10:15 - 10:35

TCCG: Tensor Contraction Code Generator

```
        for m = 1 : mc : S_{I_m}
            Â = identify_subtensor(A, m, k)
            // pack Â into Ã (L2 cache)
            Ã = packA(Â)
            Ĉ = identify_subtensor(C, m, n)
            // compute matrix-matrix product of ÃB̃
            macroKernel(Ã, B̃, Ĉ, α, β)
```

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- **Input:** Mathematical description of TC
  - e.g., `C[a,b,i,j] = A[i,k,a] * B[k,j,b];`
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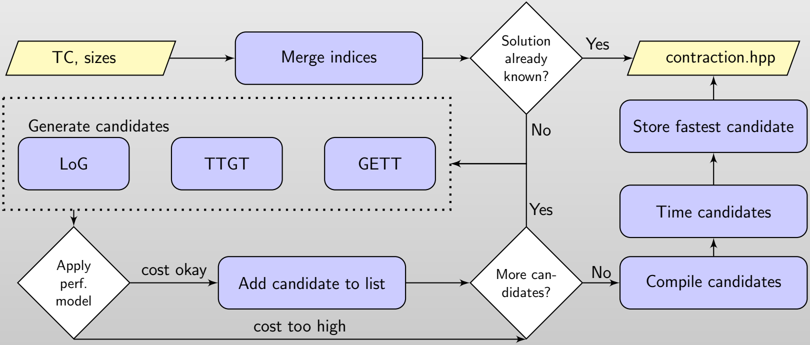
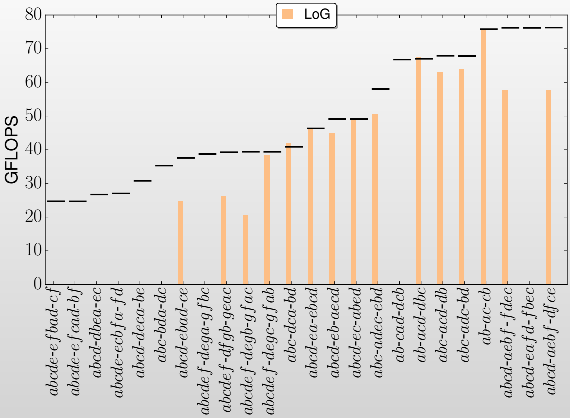


Figure: Schematic overview of TCCG.

Performance — Haswell (single core)



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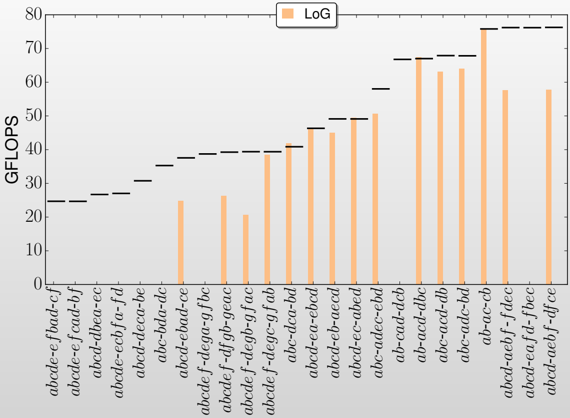

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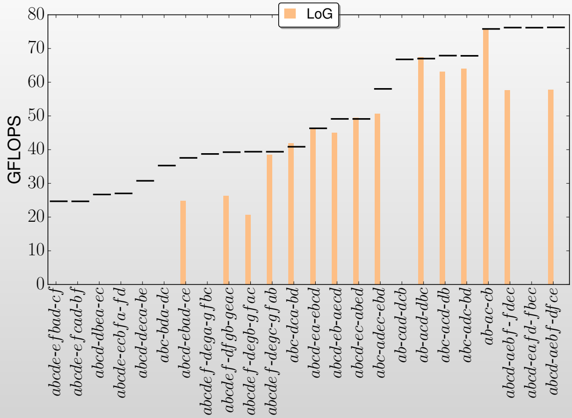
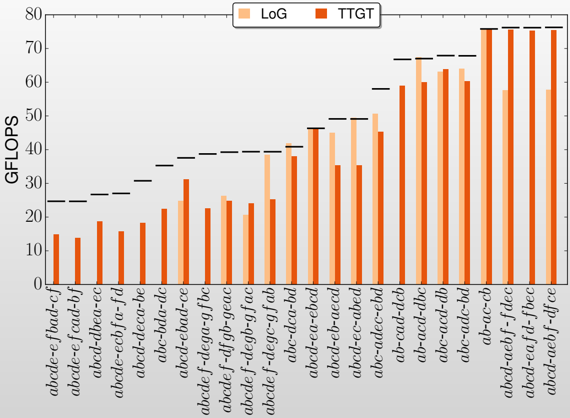
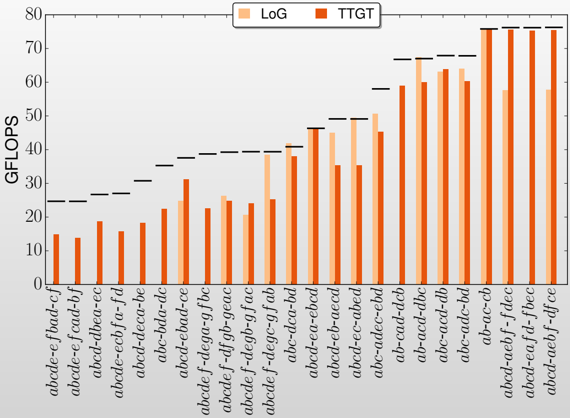


- Not all TCs can be implemented via LoG
- Mixed performance

Performance — Haswell (single core)





- TTGT: good for compute-bound TCs

Performance — Haswell (single core)

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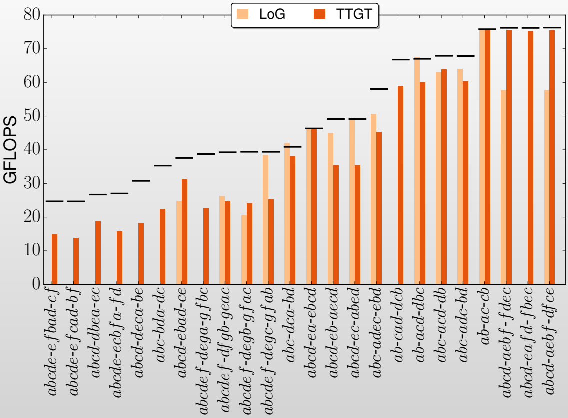
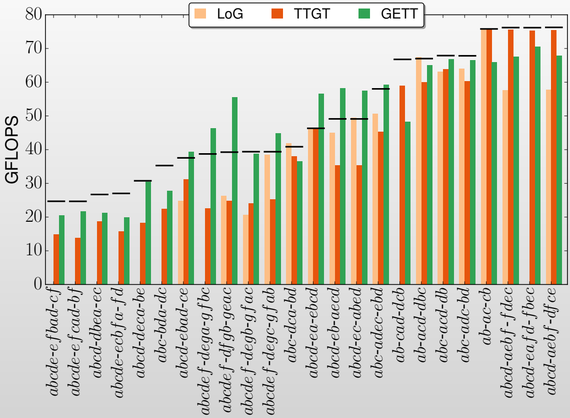


- TTGT: good for compute-bound TCs
- TTGT: bad for bandwidth-bound TCs


LoG
TTGT
GETT
GFLOPS
0
10
20
30
40
50
60
70
80
abcde-efbad-cf
abcde-efcad-bf
abcd-dbea-ec
abcde-ecbfa-fd
abcd-deca-be
abc-bda-dc
abcd-ebad-ce
abcdef-dega-gfbc
abcdef-dfgb-geac
abcdef-degb-gfac
abcdef-degc-gfab
abc-dca-bd
abcd-ea-ebcd
abcd-eb-aecd
abcd-ec-abed
abc-adec-ebd
ab-cad-dcb
ab-acd-dbc
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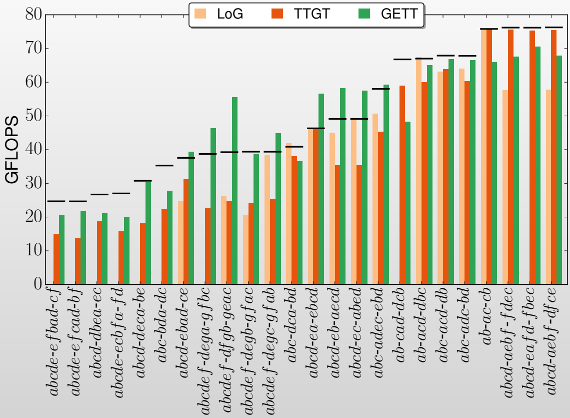

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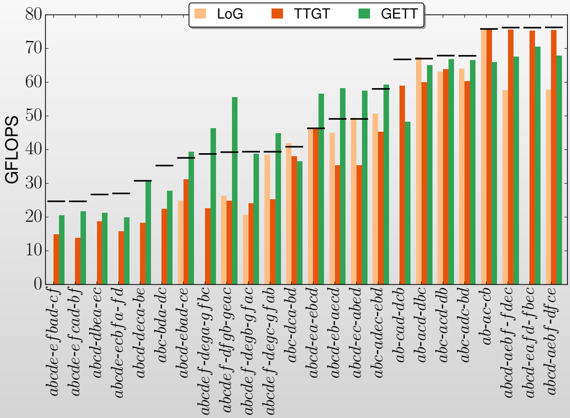

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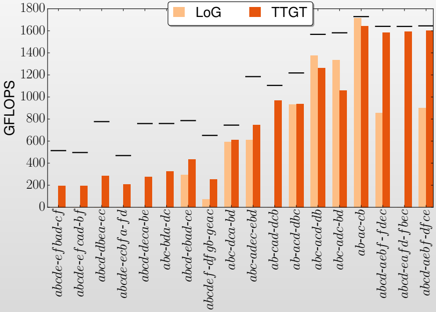
Performance — Multi-threaded



(a) 2×Intel Xeon E5-2680 v3

- Performance gap increases for bandwidth-bound TCs

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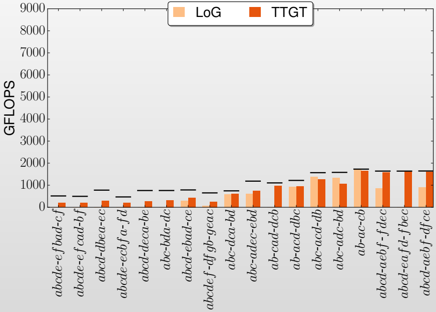

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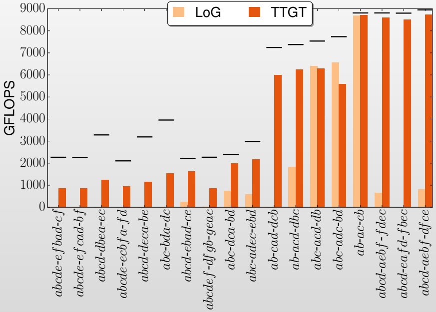

(b) NVIDIA Tesla P100

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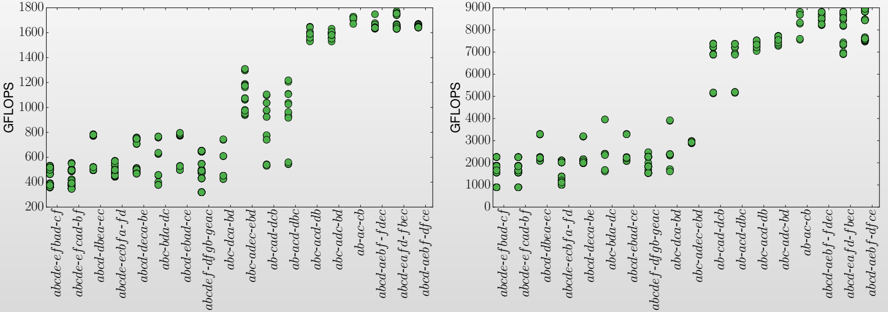
GEMM Performance — Multi-threaded



(a) 2×Intel Xeon E5-2680 v3

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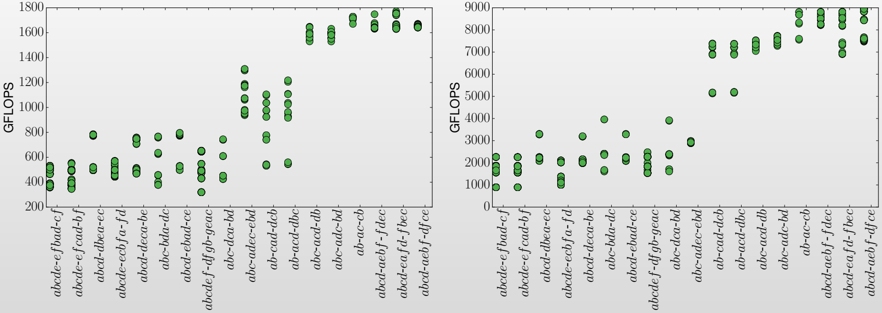
[6] Elmar Peise et al. "On the Performance Prediction of BLAS-based Tensor Contractions"

GEMM Performance — Multi-threaded



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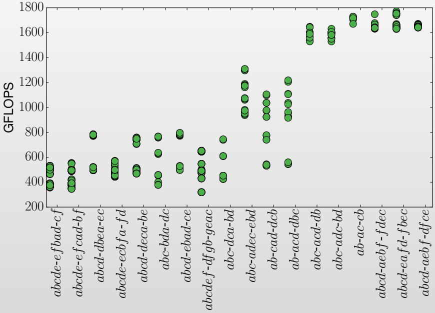
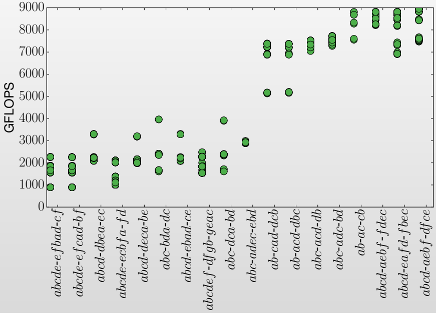
- Performance for equally-sized GEMMs varies greatly
  - For different settings: `opA`, `opB`, interchanged $\mathcal{A}$ and $\mathcal{B}$

[6] Elmar Peise et al. "On the Performance Prediction of BLAS-based Tensor Contractions"



(a) 2×Intel Xeon E5-2680 v3



(b) NVIDIA Tesla P100

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  - For different settings: `opA`, `opB`, interchanged $\mathcal{A}$ and $\mathcal{B}$
- Performance Model for TTGT and LoG:
  - Account for varying GEMM perf[6]

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Conclusion and Future Work

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- GETT exhibits high performance across a wide range of TCs
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Thank you for your attention.