Programming languages and linear algebra computations

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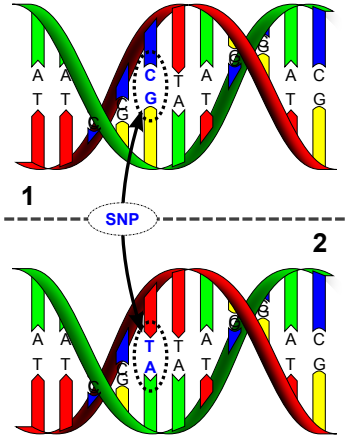
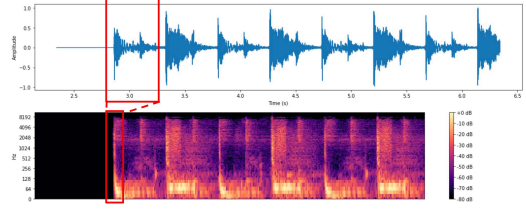
Uppsala University (via Zoom)

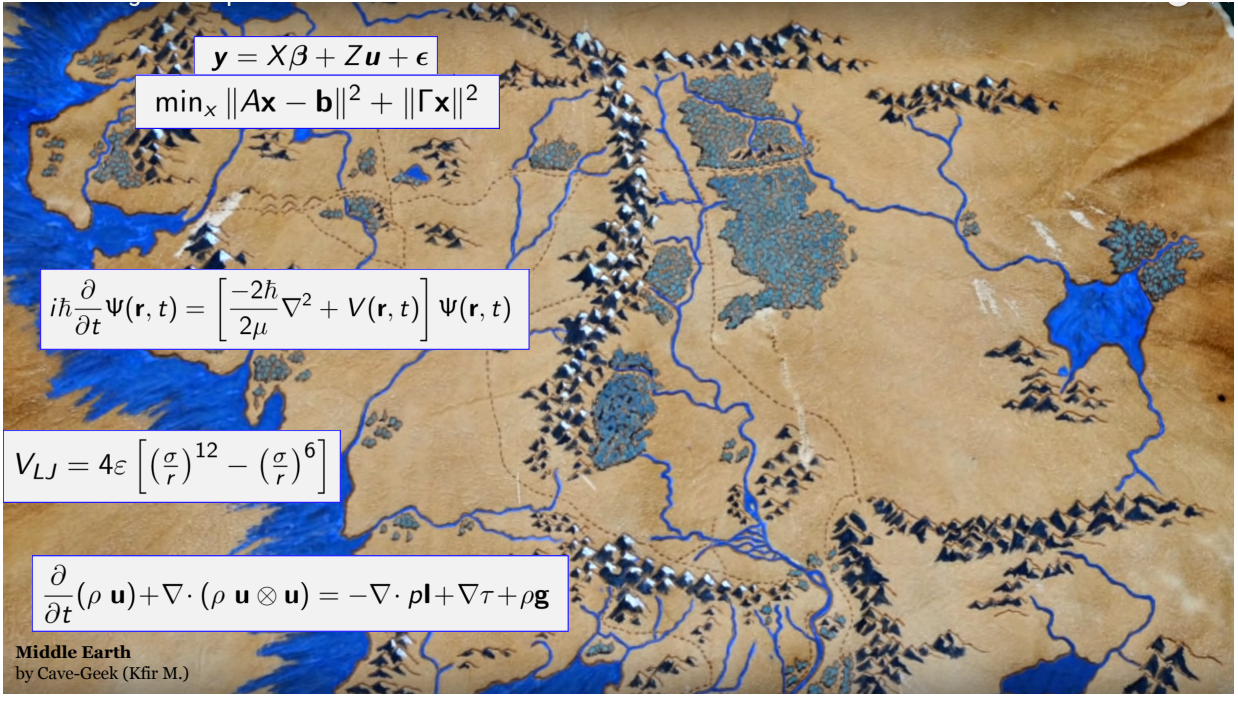
High-Performance & Scientific Computing





The computing world

Middle Earth
by Cave-Geek (Kfir M.)



Middle Earth
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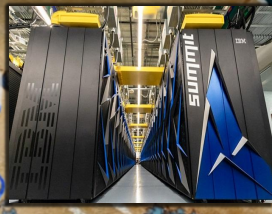
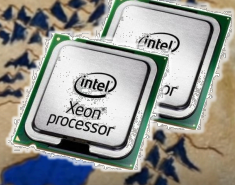
$$\boldsymbol{y} = X\boldsymbol{\beta} + Z\boldsymbol{u} + \boldsymbol{\epsilon}$$

$$\min_x \|A\mathbf{x} - \mathbf{b}\|^2 + \|\Gamma\mathbf{x}\|^2$$



$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left[\frac{-2\hbar}{2\mu}\nabla^2 + V(\mathbf{r},t)\right]\Psi(\mathbf{r},t)$$

$$V_{LJ} = 4\varepsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right]$$



$$\frac{\partial}{\partial t}(\rho\ \mathbf{u}) + \nabla\cdot(\rho\ \mathbf{u}\otimes\mathbf{u}) = -\nabla\cdot p\mathbf{I} + \nabla\tau + \rho\mathbf{g}$$

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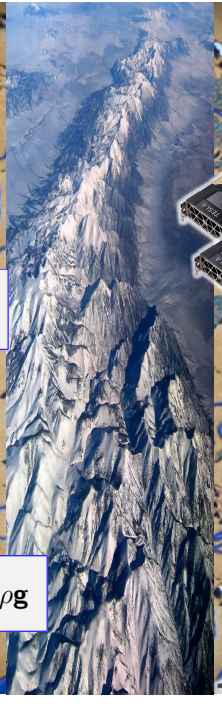

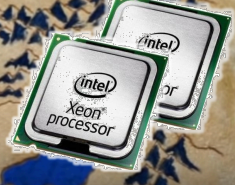

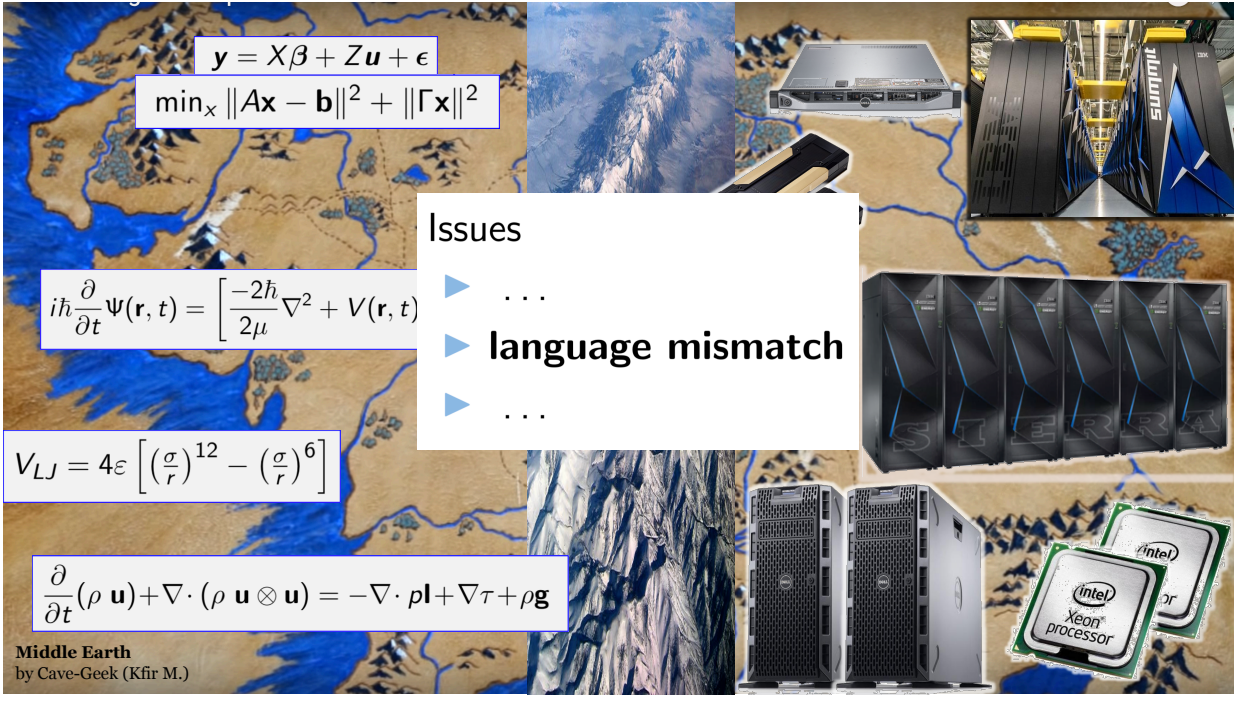
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Issues
...
language mismatch
...
Middle Earth
by Cave-Geek (Kfir M.)
Summit
SIERRA
intel
Xeon processor

This talk

matrix computations

- languages
- libraries
- compilers

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...

AND

...

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CMPSS

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Matrix computations

Signal Processing	$x := \left(A^{-T}B^{T}BA^{-1} + R^{T}LR\right)^{-1} A^{-T}B^{T}BA^{-1}y$	$R \in \mathbb{R}^{n-1\times n}$, UT; $L \in \mathbb{R}^{n-1\times n-1}$, DI
Kalman Filter	$K_k := P_k^b H^T (HP_k^b H^T + R)^{-1};\ x_k^a := x_k^b + K_k(z_k - Hx_k^b);\ P_k^a := (I - K_K H)\, P_k^b$	
Ensemble Kalman Filter	$X^a := X^b + \left(B^{-1} + H^T R^{-1} H\right)^{-1} \left(Y - HX^b\right)$	$B \in \mathbb{R}^{N\times N}$ SSPD; $R \in \mathbb{R}^{m\times m}$, SSPD
Ensemble Kalman Filter	$\delta X := \left(B^{-1} + H^T R^{-1} H\right)^{-1} H^T R^{-1} \left(Y - HX^b\right)$	
Ensemble Kalman Filter	$\delta X := XV^T \left(R + HX(HX)^T\right)^{-1} \left(Y - HX^b\right)$	
Image Restoration	$x_k := (H^T H + \lambda\sigma^2 I_n)^{-1}(H^T y + \lambda\sigma^2(v_{k-1} - u_{k-1}))$	
Image Restoration	$H^\dagger := H^T(HH^T)^{-1};\ y_k := H^\dagger y + (I_n - H^\dagger H)x_k$	
Rand. Matrix Inversion	$X_{k+1} := S(S^TAS)^{-1}S^T + (I_n - S(S^TAS)^{-1}S^TA)X_k(I_n - AS(S^TAS)^{-1}S^T)$	
Rand. Matrix Inversion	$X_{k+1} := X_k + WA^TS(S^TAWA^TS)^{-1}S^T(I_n - AX_k)$	$W \in \mathbb{R}^{n\times n}$, SPD
Rand. Matrix Inversion	$X_{k+1} := X_k + (I_n - X_kA^T)S(S^TA^TWAS)^{-1}S^TA^TW$	
Rand. Matrix Inversion	$\Lambda := S(S^TAWAS)^{-1}S^T;\ \Theta := \Lambda AW;\ M_k := X_kA - I$ $X_{k+1} := X_k - M_k\Theta - (M_k\Theta)^T + \Theta^T(AX_kA - A)\Theta$	

Matrix computations (2)

Generalized Least Squares	$b := (X^T M^{-1} X)^{-1} X^T M^{-1} y$	$n > m$; $M \in \mathbb{R}^{n \times n}$, SPD; $X \in \mathbb{R}^{n \times m}$; $y \in \mathbb{R}^{n \times 1}$
Stochastic Newton	$B_k := \frac{k}{k-1} B_{k-1}(I_n - A^T W_k((k-1)I_l + W_k^T A B_{k-1} A^T W_k)^{-1} W_k^T A B_{k-1})$	
Optimization	$x_f := WA^T(AWA^T)^{-1}(b - Ax)$; $x_o := W(A^T(AWA^T)^{-1}Ax - c)$	
Optimization	$x := W(A^T(AWA^T)^{-1}b - c)$	
Triangular Matrix Inv.	$X_{10} := L_{10}L_{00}^{-1}$; $X_{20} := L_{20} + L_{22}^{-1}L_{21}L_{11}^{-1}L_{10}$; $X_{11} := L_{11}^{-1}$; $X_{21} := -L_{22}^{-1}L_{21}$	
Tikhonov Regularization	$x := (A^T A + \Gamma^T \Gamma)^{-1} A^T b$	$A \in \mathbb{R}^{n \times m}$; $\Gamma \in \mathbb{R}^{m \times m}$; $b \in \mathbb{R}^{n \times 1}$
Tikhonov Regularization	$x := (A^T A + \alpha^2 I)^{-1} A^T b$	
Gen. Tikhonov Reg.	$x := (A^T PA + Q)^{-1}(A^T Pb + Qx_0)$	$P \in \mathbb{R}^{n \times n}$, SSPD; $Q \in \mathbb{R}^{m \times m}$, SSPD; $x_0 \in \mathbb{R}^{m \times 1}$
Gen. Tikhonov reg.	$x := x_0 + (A^T PA + Q)^{-1}(A^T P(b - Ax_0))$	
LMMSE estimator	$K_{t+1} := C_t A^T (AC_t A^T + C_z)^{-1}$; $x_{t+1} := x_t + K_{t+1}(y - Ax_t)$; $C_{t+1} := (I - K_{t+1}A)C_t$	
LMMSE estimator	$x_{\text{out}} = C_X A^T (AC_X A^T + C_Z)^{-1}(y - Ax) + x$	
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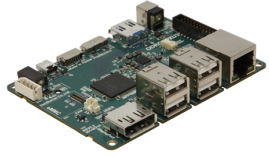

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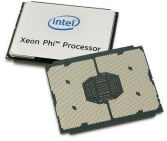

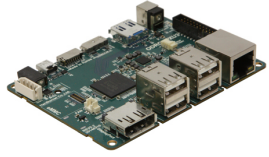

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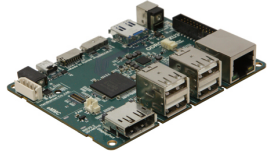
2-step solution



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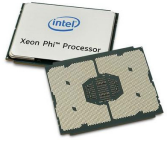
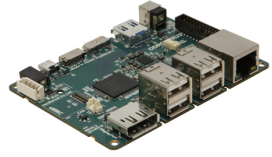
building blocks (libraries)





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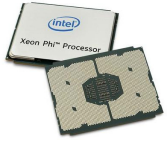

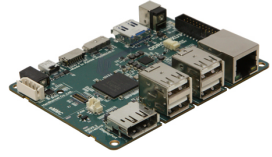

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- **Libraries**: necessity (wrt performance)

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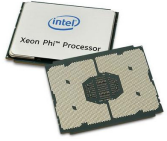
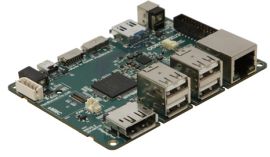
... **BLAS** **LAPACK** ...





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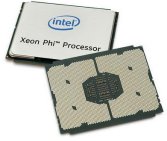

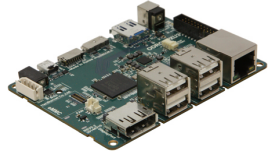

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$$\begin{cases} C_\dagger := PCP^T + Q \\ K := C_\dagger H^T (H C_\dagger H^T)^{-1} \end{cases}$$

$$\Lambda := S(S^T A W A S)^{-1} S^T; \ \Theta := \Lambda A W; \ M_k := X_k A - I$$
$$X_{k+1} := X_k - M_k \Theta - (M_k \Theta)^T + \Theta^T (A X_k A - A) \Theta$$

$$x := A(B^T B + A^T R^T \Lambda R A)^{-1}$$

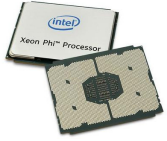
$$= Q^{-1} U (I + U^T Q^{-1} U)^{-1} U^T$$

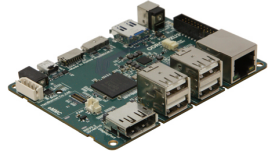
Step 2: "LAMP"
LINEAR ALGEBRA MAPPING PROBLEM

$y := \alpha x + y$ | $LU = A$ | $\cdots$ | $C := \alpha AB + \beta C$

$X := A^{-1} B$ | $C := AB^T + BA^T + C$ | $X := L^{-1} M L^{-T}$ | $QR = A$

... **BLAS** **LAPACK** ...





MUL | ADD | MOV

MOVAPD

VFMADDPD ...

Linear Algebra Mapping Problem

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- $\mathcal{E}$: a sequence of explicit assignments $var_i := EXP_i$

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Find a sequence of calls to building blocks in $\mathcal{K}$, optimal according to $\mathcal{M}$, that computes all the assignments in $\mathcal{E}$.

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LAMP:
Find a sequence of calls to building blocks in $\mathcal{K}$, optimal according to $\mathcal{M}$, that computes all the assignments in $\mathcal{E}$.

- Suboptimal solution — easy
- Optimality — NP complete — reduction from Ensemble Computation

Problem acknowledged, yet overlooked

"How do users go about solving their matrix computations?"

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Slow solutions → "2-language problem"

Investigation: How well do high-level languages solve LAMPs?

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```
C = A * B'+ B * A' + C;                          // Matlab, Octave

C = A * transpose(B) + B * transpose(A) + C      // Julia

C = A * trans(B) + B * trans(A) + C;             // Armadillo

C = A * B.transpose() + B * A.transpose() + C;   // Eigen

ct = at @ bt.T + bt @ at.T + ct                  // NumPy

ct <- at %*% t(bt) + bt %*% t(at) + ct           // R
```

Do they map? matrix products

	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
$C = AB$								

Do they map? matrix products

	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
$C = AB$	0.29	0.28	0.30	0.31	0.29	0.29	0.29	0.27

Do they map? matrix products

	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
$C = AB$	0.29	0.28	0.30	0.31	0.29	0.29	0.29	0.27
`GEMM`	✓	✓	✓	✓	✓	✓	✓	

Do they map? matrix products

	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
C = AB	0.29	0.28	0.30	0.31	0.29	0.29	0.29	0.27
`GEMM`	✓	✓	✓	✓	✓	✓	✓	

C = C+AA'

Do they map? matrix products

	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
C = AB	0.29	0.28	0.30	0.31	0.29	0.29	0.29	0.27
`GEMM`	✓	✓	✓	✓	✓	✓	✓	
C = C+AA'	0.18	0.17	0.21	0.32	0.29	0.17	0.18	0.14

Do they map? matrix products

	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
C = AB	0.29	0.28	0.30	0.31	0.29	0.29	0.29	0.27
`GEMM`	✓	✓	✓	✓	✓	✓	✓	
C = C+AA'	0.18	0.17	0.21	0.32	0.29	0.17	0.18	0.14
`SYRK`	✓	✓	✓	×	×	✓	✓	

Do they map? matrix products

	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
C = AB	0.29	0.28	0.30	0.31	0.29	0.29	0.29	0.27
`GEMM`	✓	✓	✓	✓	✓	✓	✓	
C = C+AA'	0.18	0.17	0.21	0.32	0.29	0.17	0.18	0.14
`SYRK`	✓	✓	✓	×	×	✓	✓	
C = C+AB'+BA'	0.57	0.59	0.69	0.59	0.58	0.57	0.58	0.28

Do they map? matrix products

	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
C = AB	0.29	0.28	0.30	0.31	0.29	0.29	0.29	0.27
`GEMM`	✓	✓	✓	✓	✓	✓	✓	
C = C+AA'	0.18	0.17	0.21	0.32	0.29	0.17	0.18	0.14
`SYRK`	✓	✓	✓	×	×	✓	✓	
C = C+AB'+BA'	0.57	0.59	0.69	0.59	0.58	0.57	0.58	0.28
`SYR2K`	×	×	×	×	×	×	×	

Do they map? $Ax = b \quad \equiv \quad x := A\backslash b \quad \not\equiv \quad \text{inv}(A) * b$

	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
$x := A\backslash b$								

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	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
$x := A\backslash b$	0.71	0.72	0.63	0.68	0.64	0.63	0.72	0.61

Do they map? $Ax = b \quad \equiv \quad x := A\backslash b \quad \not\equiv \quad \text{inv}(A) * b$

	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
$x := A\backslash b$	0.71	0.72	0.63	0.68	0.64	0.63	0.72	0.61
$\text{inv}(A) * b$	1.76	1.82	1.69	2.20	2.21	**0.63**	2.49	1.71

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LinSolve	-	-	-	-	-	✓	-	-

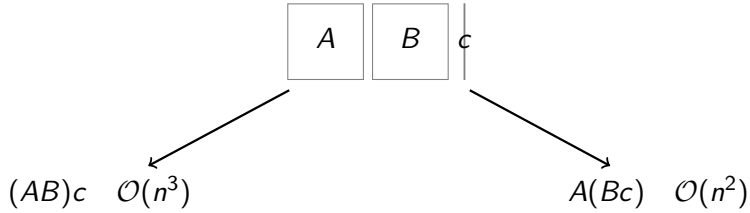
but ...

Do they map? $Ax = b \quad \equiv \quad x := A\backslash b \quad \not\equiv \quad \text{inv}(A) * b$

	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
$x := A\backslash b$	0.71	0.72	0.63	0.68	0.64	0.63	0.72	0.61
$\text{inv}(A) * b$	1.76	1.82	1.69	2.20	2.21	**0.63**	2.49	1.71
LinSolve	-	-	-	-	-	✓	-	-

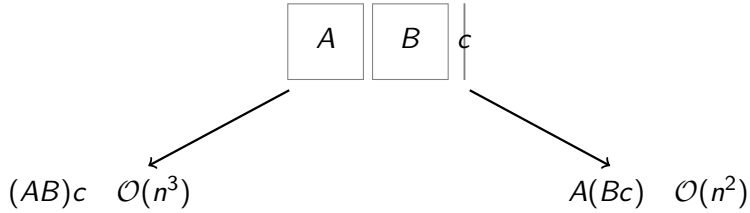
but . . . should they map?

Optimal parenthesisation



Product is associative, but its cost is not

Optimal parenthesisation



Product is associative, but its cost is not

Matrix Chain Algorithm $O(k \log k)$ Hu & Shing 1982; $O(k^3)$ dynamic programming

Matrix Chain?

Chain	**Optimal Evaluation**
1) "left-to-right" (LtR)	((A B) C)
2) "right-to-left" (RtL)	(A (B C))
3) "mixed" (Mix)	((A B) (C D))

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		Matlab	Octave	Julia	R	Eigen	Armad.	NumPy
1)	LtR no par.	0.056	0.056	0.055	0.061	0.058	0.056	0.055
	LtR guided	0.056	0.056	0.055	0.061	0.058	0.056	0.055

Matrix Chain?

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1) "left-to-right" (LtR)	((A B) C)
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3) "mixed" (Mix)	((A B) (C D))

		Matlab	Octave	Julia	R	Eigen	Armad.	NumPy
1)	LtR no par.	0.056	0.056	0.055	0.061	0.058	0.056	0.055
	LtR guided	0.056	0.056	0.055	0.061	0.058	0.056	0.055
2)	RtL no par.	0.42	0.43	0.42	0.44	0.42	**0.055**	0.42
	RtL guided	0.055	0.056	0.054	0.059	0.056	0.055	0.056

Matrix Chain?

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1) "left-to-right" (LtR)	((A B) C)
2) "right-to-left" (RtL)	(A (B C))
3) "mixed" (Mix)	((A B) (C D))

		Matlab	Octave	Julia	R	Eigen	Armad.	NumPy
1)	LtR no par.	0.056	0.056	0.055	0.061	0.058	0.056	0.055
	LtR guided	0.056	0.056	0.055	0.061	0.058	0.056	0.055
2)	RtL no par.	0.42	0.43	0.42	0.44	0.42	**0.055**	0.42
	RtL guided	0.055	0.056	0.054	0.059	0.056	0.055	0.056
3)	Mix no par.	0.32	0.33	0.33	0.33	0.35	0.31	0.33
	Mix guided	0.21	0.22	0.22	0.22	0.23	0.20	0.22

Matrix Chain?

Chain	**Optimal Evaluation**
1) "left-to-right" (LtR)	((A B) C)
2) "right-to-left" (RtL)	(A (B C))
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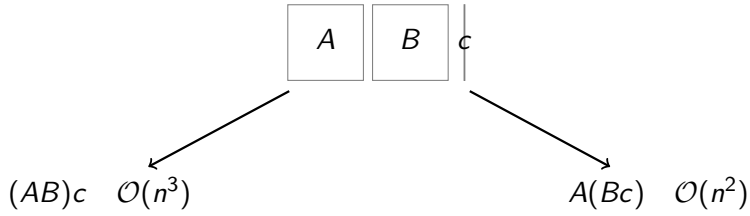
		Matlab	Octave	Julia	R	Eigen	Armad.	NumPy
1)	LtR no par.	0.056	0.056	0.055	0.061	0.058	0.056	0.055
	LtR guided	0.056	0.056	0.055	0.061	0.058	0.056	0.055
2)	RtL no par.	0.42	0.43	0.42	0.44	0.42	**0.055**	0.42
	RtL guided	0.055	0.056	0.054	0.059	0.056	0.055	0.056
3)	Mix no par.	0.32	0.33	0.33	0.33	0.35	0.31	0.33
	Mix guided	0.21	0.22	0.22	0.22	0.23	0.20	0.22
	Matrix chains	×	×	×	×	×	≈	×

Challenge: Generalized Matrix Chain

[arXiv:1804.04021]



In practice

- Unary operators: transposition, inversion $(X := AB^T C^{-T} D + \dots)$
- Overlapping kernels (e.g., $L \leftarrow L^{-1}$, $X = A^{-1}B$)
- Decompositions (e.g., $A \rightarrow Q^T D Q$, $A \rightarrow LU$)
- Properties & specialized kernels (`GEMM`, `TRMM`, `SYMM`, ...)

Challenge: Not all flops were created equal

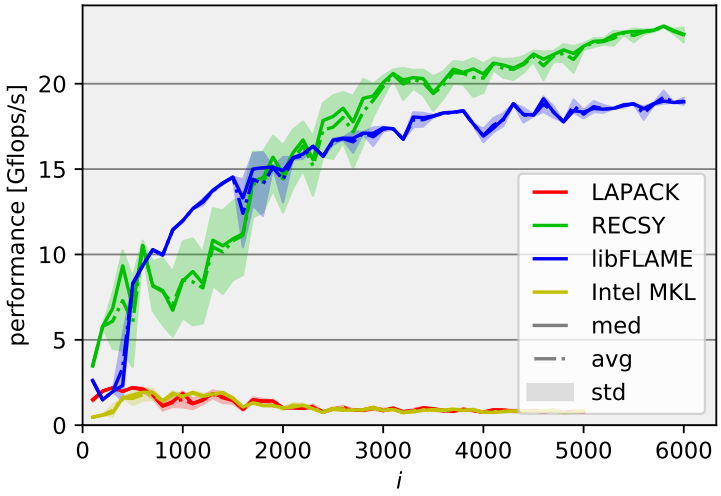
$$\#\text{FLOPs} \quad \text{vs.} \quad \text{execution time} \qquad (\text{vs. numerical stability})$$

$$\underset{\mathcal{A}}{\operatorname{argmin}} \left(\ \text{FLOPs}(\mathcal{A}) \ \right) \neq \underset{\mathcal{A}}{\operatorname{argmin}} \left(\ \text{time}(\mathcal{A}) \ \right)$$

Challenge: Not all flops were created equal

[arXiv:1504.08035]

Triangular Sylvester equation – all libraries perform the same # of flops



Challenge: Not all flops were created equal

[arXiv:1706.01341]

#FLOPs vs. execution time (vs. numerical stability)

$$\operatorname*{argmin}_{\mathcal{A}} \left(\text{FLOPs}(\mathcal{A}) \right) \neq \operatorname*{argmin}_{\mathcal{A}} \left(\text{time}(\mathcal{A}) \right)$$

$\Rightarrow$ **Performance prediction**: efficiency

Challenge: Not all flops were created equal [arXiv:1706.01341]

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$\Rightarrow$ **Performance prediction**: efficiency

Big Challenge: **Parallelism**

$$X := A((B^T C^{-T})D) \quad \text{vs.} \quad X := (AB^T)(C^{-T}D) \quad \text{vs.} \quad \dots$$

Properties?

Operation	Property	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
Linear System	-	0.71	0.74	0.62	0.67	0.63	0.62	0.65	0.61

Properties?

Operation	Property	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
Linear System	-	0.71	0.74	0.62	0.67	0.63	0.62	0.65	0.61
	Symmetric	0.71	0.73	0.62	0.69	N/A	0.62	0.65	0.46

Properties?

Operation	Property	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
Linear System	-	0.71	0.74	0.62	0.67	0.63	0.62	0.65	0.61
	Symmetric	0.71	0.73	0.62	0.69	N/A	0.62	0.65	0.46
	SPD	**0.41**	**0.40**	0.60	0.63	N/A	**0.34**	0.62	0.31

Properties?

Operation	Property	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
Linear System	-	0.71	0.74	0.62	0.67	0.63	0.62	0.65	0.61
	Symmetric	0.71	0.73	0.62	0.69	N/A	0.62	0.65	0.46
	SPD	**0.41**	**0.40**	0.60	0.63	N/A	**0.34**	0.62	0.31
	Triangular	**0.03**	**0.04**	**0.03**	0.63	N/A	0.62	0.65	0.03

Properties?

Operation	Property	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
Linear System	-	0.71	0.74	0.62	0.67	0.63	0.62	0.65	0.61
	Symmetric	0.71	0.73	0.62	0.69	N/A	0.62	0.65	0.46
	SPD	**0.41**	**0.40**	0.60	0.63	N/A	**0.34**	0.62	0.31
	Triangular	**0.03**	**0.04**	**0.03**	0.63	N/A	0.62	0.65	0.03
	Diagonal	0.03	0.05	**0.01**	0.63	N/A	**0.03**	0.62	0.001
		$\approx$	$\approx$	$\approx$	$\times$	$\times$	$\approx$	$\times$	

Properties?

Operation	Property	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy	C
Linear System	-	0.71	0.74	0.62	0.67	0.63	0.62	0.65	0.61
	Symmetric	0.71	0.73	0.62	0.69	N/A	0.62	0.65	0.46
	SPD	**0.41**	**0.40**	0.60	0.63	N/A	**0.34**	0.62	0.31
	Triangular	**0.03**	**0.04**	**0.03**	0.63	N/A	0.62	0.65	0.03
	Diagonal	0.03	0.05	**0.01**	0.63	N/A	**0.03**	0.62	0.001
		$\approx$	$\approx$	$\approx$	$\times$	$\times$	$\approx$	$\times$	
Multiplication	-	1.44	1.48	1.47	1.47	1.45	1.44	1.44	1.46
	Triangular	1.44	1.48	**0.75**	1.47	1.45	1.44	1.44	0.74
	Diagonal	1.44	1.48	**0.03**	1.47	1.45	1.42	1.44	0.06
		$\times$	$\times$	✓	$\times$	$\times$	$\times$	$\times$	

Challenge: Inference of properties

- **easy** $E := L_1 * U^T * L_2$ triangular(E) ?

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- **impossible?** $E := Q^{-1}U(I + U^TQ^{-1}U)^{-1}U^T$ properties($I + U^TQ^{-1}U$) ?

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$\Rightarrow$ **Symbolic analysis**: pattern matching

Challenge: Common Subexpressions

[arXiv:1710.06915]

$$\begin{cases} X := AB^{-T}C \\ Y := B^{-1}A^{T}D \end{cases} \quad \rightarrow \quad \begin{cases} Z := AB^{-T} \\ X := ZC \\ Y := Z^{T}D \end{cases}$$

Challenge: Common Subexpressions

[arXiv:1710.06915]

$$\begin{cases} X := AB^{-T}C \\ Y := B^{-1}A^{T}D \end{cases} \quad \rightarrow \quad \begin{cases} Z := AB^{-T} \\ X := ZC \\ Y := Z^{T}D \end{cases}$$

BUT

$$X := ABABv \quad \not\rightarrow \quad \begin{cases} Z := AB \\ X := ZZv \end{cases}$$

Common Subexpressions?

$$\begin{cases} X := AB \\ Y := AB \end{cases} \quad \rightarrow \quad \begin{cases} X := AB \\ Y := X \end{cases}$$

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	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy
copy	0.27	0.31	0.36	0.30	0.30	0.26	0.30

Common Subexpressions?

$$\begin{cases} X := AB \\ Y := AB \end{cases} \quad \rightarrow \quad \begin{cases} X := AB \\ Y := X \end{cases}$$

	Matlab	Octave	Julia	R	Eigen	Armad.	NumPy
copy	0.27	0.31	0.36	0.30	0.30	0.26	0.30
direct	0.54	0.6	0.61	0.56	0.58	0.52	0.55
	×	×	×	×	×	×	×

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for i = 1:n,
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- Blocked operands

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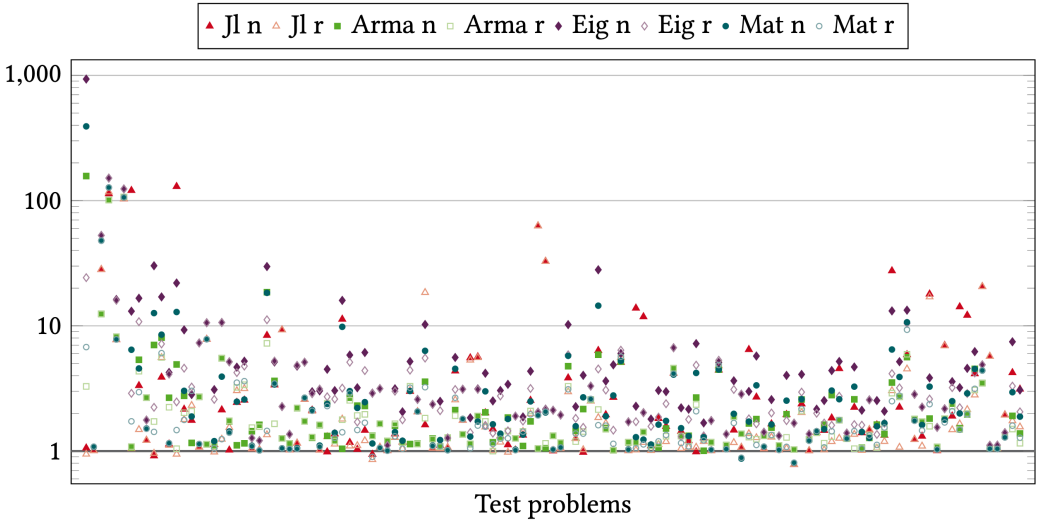
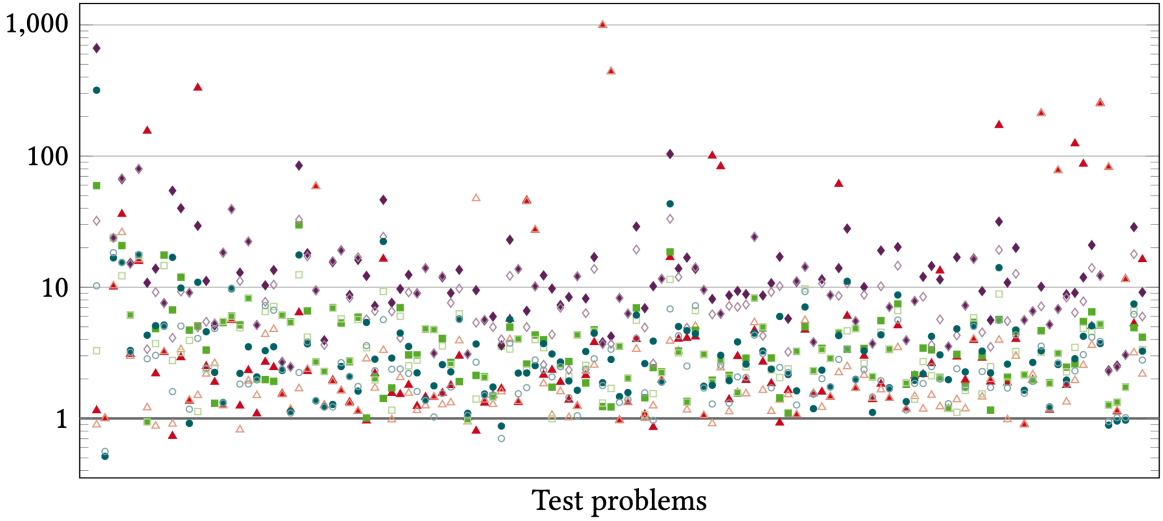
THANK YOU

Linnea speedup: 1 thread



Jl: Julia, **Arma**: Armadillo, **Eig**: Eigen, **Mat**: Matlab. **n**/**r**: naive/recommended implementation

Linnea speedup: 24 threads



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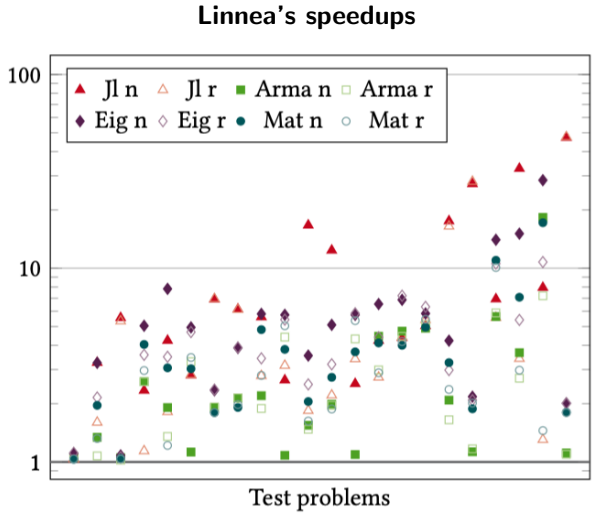
Linnea

Linear algebra knowledge: operators, identities, properties, theorems

- Distributivity, commutativity, partitionings, ...
- $((QR)^T QR)^{-1}(QR)^T y \;\rightarrow\; (R^T Q^T QR)^{-1} R^T Q^T y \;\rightarrow\; R^{-1}R^{-T}R^T Q^T y \;\rightarrow\; R^{-1}Q^T y$
- $\mathrm{SPD}(A) \rightarrow \mathrm{SPD}(A_{BR} - A_{BL}A_{TL}^{-1}A_{BL}^T)$ Schur complement
- ...

Linnea: Results for real applications



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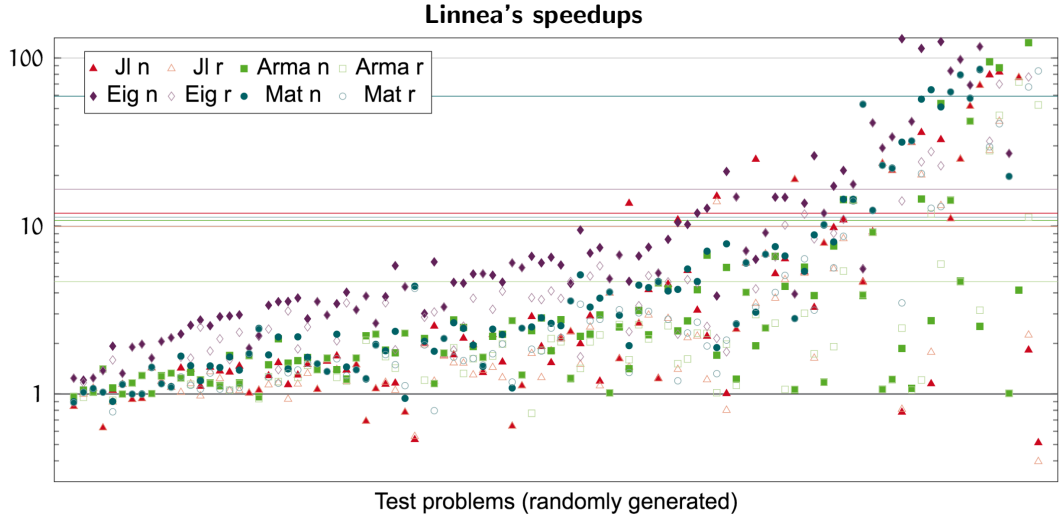
Linnea: Results for random expressions



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- Applications

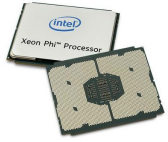
  $\mathcal{E}$: explicit assignments $\mathcal{K}$: libraries $\mathcal{M}$: execution time + ...

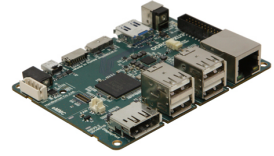
Quantum Chemistry | High-order FEM | DG | ... | Tensor App #N

Transposition | Contraction | ... | Alternating LS

Khatri-Rao | SpMTTKRP | ... | TTV, TTM

HPTT **TCL** **...** **BLAS**





MUL | ADD | MOV

MOVAPD

VFMADDPD ...

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$$w := AB^{-1}c, \quad \text{SPD}(B)$$

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w = A*(B\c)
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Generated

```
ml0 = A; ml1 = B; ml2 = c;
potrf!('L', ml1)
trsv!('L', 'N', 'N', ml1, ml2)
trsv!('L', 'T', 'N', ml1, ml2)
ml3 = Array{Float64}(10)
gemv!('N', 1.0, ml0, ml2, 0.0, ml3)
w = ml3
```